On the number of Courant-sharp Dirichlet eigenvalues

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Abstract

We consider arbitrary open sets $\Omega$ in Euclidean space with finite Lebesgue measure, and obtain upper bounds for (i) the largest Courant-sharp Dirichlet eigenvalue of $\Omega$, (ii) the number of Courant-sharp Dirichlet eigenvalues of $\Omega$. This extends recent results of P. Bérard and B. Helffer.

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1 Introduction

Let $\Omega$ be an open set in Euclidean space $\mathbb{R}^m$ with finite Lebesgue measure $|\Omega|$ and boundary $\partial \Omega$. We denote the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ by $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \ldots$ taking the multiplicities of these eigenvalues into account. We define the counting function for $\Omega$ by

$$N_\Omega(\lambda) = \sharp \{ n \in \mathbb{N} : \lambda_n(\Omega) < \lambda \}.$$

Weyl’s law asserts that

$$N_\Omega(\lambda) = \frac{\omega_m}{(2\pi)^m} |\Omega| \lambda^{m/2} + o(\lambda^{m/2}), \quad \lambda \rightarrow \infty,$$  \hfill (1)

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where $\omega_m$ is the measure of a ball $B_m$ with radius 1 in $\mathbb{R}^m$. We refer to Theorem 2 in [16] for a proof of (1) in this generality. For a proof of Weyl’s law with a non-trivial remainder estimate for $\Omega$ open, bounded and connected we refer to Theorem 1.8 in [12].

Let $\{\varphi_1, \varphi_2, \ldots\}$ be an orthonormal basis in the Sobolev space $H^1_0(\Omega)$ of eigenfunctions corresponding to the Dirichlet eigenvalues. These eigenfunctions satisfy the Dirichlet boundary conditions in the usual trace sense. Let $\nu(\varphi_n, \Omega)$ denote the number of nodal domains of $\varphi_n, \Omega$. Then Pleijel’s theorem ([13]) states that

$$\limsup_{n \to \infty} \frac{\nu(\varphi_n, \Omega)}{n} \leq \gamma_m,$$

where

$$\gamma_m = \frac{(2\pi)^m}{\omega_m^2} \left( \lambda_1(B_m) \right)^{-m/2} < 1.$$  

(2)

It is known that Pleijel’s bound is not sharp. See [7], [18] and [14].

We say that $\lambda_n(\Omega)$ is Courant-sharp if $\nu(\varphi_n, \Omega) = n$. Courant’s nodal domain theorem asserts that $\nu(\varphi_n, \Omega) \leq n$. Courant’s original proof in [8] was for the planar case. This has been subsequently stated and proved in a Riemannian manifold setting in [3]. See also [13]. Pleijel’s theorem implies that for a given $\Omega$ the number of Courant-sharp Dirichlet eigenvalues is finite. Using results of [5] and [17], Bérad and Helffer, [1], obtained an upper bound for the largest Courant-sharp Dirichlet eigenvalue if $\Omega$ is bounded and has smooth boundary $\partial\Omega$.

This paper concerns arbitrary open sets in $\mathbb{R}^m$ with finite Lebesgue measure. The proofs of Courant’s theorem in [8], [13] and [3] all use the fact that a restriction of an eigenfunction to a nodal domain $U$ is the first Dirichlet eigenfunction on $U$. This has been subsequently stated and proved in a Riemannian manifold setting in [3]. See also [13]. Pleijel’s theorem implies that for a given $\Omega$ the number of Courant-sharp Dirichlet eigenvalues is finite. Using results of [5] and [17], Bérad and Helffer, [1], obtained an upper bound for the largest Courant-sharp Dirichlet eigenvalue if $\Omega$ is bounded and has smooth boundary $\partial\Omega$.

Our main result, Theorem 1 below is for open sets $\Omega$ in $\mathbb{R}^m$ with finite Lebesgue measure. We obtain (i) an upper bound for the largest Dirichlet eigenvalue of $\Omega$ which is Courant-sharp, and (ii) an upper bound for the number of Courant-sharp eigenvalues of $\Omega$. For $A \subset \mathbb{R}^m, A \neq \emptyset$ let

$$d(x, A) = \inf \{|x - y| : y \in A\}.$$  

For $\epsilon \geq 0$ and $|\Omega| < \infty$ we define

$$\mu_\Omega(\epsilon) = |\{x \in \Omega : d(x, \partial\Omega) < \epsilon\}|,$$

and

$$\epsilon(\Omega) = \inf \{\epsilon : \mu_\Omega(\epsilon) \geq 2^{-1}(1 - \gamma_m)|\Omega|\}.$$  

(3)

We denote the number of Courant-sharp eigenvalues of $\Omega$ by $C(\Omega)$.

**Theorem 1.** Let $\Omega$ be an open set in $\mathbb{R}^m$ with finite Lebesgue measure. We have the following.

(i) If $\lambda_n(\Omega)$ is Courant-sharp then

$$\lambda_n(\Omega) \leq \left( \frac{2\pi m^2}{(1 - \gamma_m)\epsilon(\Omega)} \right)^2.$$  

(4)
\[ \mathcal{C}(\Omega) \leq \frac{\omega_m}{(1 - \gamma_m)^m} \left( m^3(m + 2) \right)^{m/2} \frac{|\Omega|}{\epsilon(\Omega)^m}. \]  

(iii) If \( n \in \mathbb{N}, n > \frac{\omega_m}{(1 - \gamma_m)^m} \left( m^3(m + 2) \right)^{m/2} \frac{|\Omega|}{\epsilon(\Omega)^m} \), then \( \lambda_n(\Omega) \) is not Courant-sharp.

In Section 2 below we prove Theorem 1. In Section 3 we analyse some examples including the von Koch snowflake.

2 Proof of Theorem 1

Suppose \( \lambda_n(\Omega) \) is Courant-sharp with eigenfunction \( \varphi_{n,\Omega} \). Let \( U_1, \ldots, U_n \) be the nodal domains of \( \varphi_{n,\Omega} \) so that \( \lambda_n(\Omega) = \lambda_1(U_1) = \cdots = \lambda_1(U_n) \). Without loss of generality we may assume that \( |U_1| \leq |U_2| \leq \cdots \leq |U_n| \). Hence \( |U_1| \leq |\Omega|/n \).

By Faber-Krahn we have that
\[ \lambda_n(\Omega) = \lambda_1(U_1) \geq \lambda_1(B_m) \left( \frac{n\omega_m}{|\Omega|} \right)^{2/m}. \]

It follows that, since \( \lambda_{n-1}(\Omega) < \lambda_n(\Omega) \),
\[ (\lambda_n(\Omega))^{m/2} \geq (\lambda_1(B_m))^{m/2} \frac{n\omega_m}{|\Omega|} \]
\[ \geq (\lambda_1(B_m))^{m/2} \frac{\omega_m}{|\Omega|} (n - 1) \]
\[ = (\lambda_1(B_m))^{m/2} \frac{\omega_m}{|\Omega|} N_\Omega(\lambda_n(\Omega)). \]

This gives that
\[ \frac{\omega_m}{(2\pi)^m} (1 - \gamma_m)|\Omega| (\lambda_n(\Omega))^{m/2} \leq R_\Omega(\lambda_n(\Omega)), \]
where \( R_\Omega : \mathbb{R}^+ \rightarrow \mathbb{R} \) is defined by
\[ R_\Omega(\lambda) = \frac{\omega_m}{(2\pi)^m} |\Omega| \lambda^{m/2} - N_\Omega(\lambda). \]

See (15) and (16) in [1]. Below we obtain an upper bound for \( R_\Omega(\lambda) \). Let \( \epsilon > 0 \) be arbitrary. Consider the collection \( \mathfrak{M}_\epsilon \) of open cubes of measure \( \epsilon^m \) with vertices in the set of \( m \)-tuples \( \{Z\epsilon, \ldots, Z\epsilon\} \). Let \( M_\Omega(\epsilon) \) be the number of open cubes of side-length \( \epsilon \) in \( \mathfrak{M}_\epsilon \) which are contained in \( \Omega \),
\[ M_\Omega(\epsilon) = \sharp \{ N \in \mathfrak{M}_\epsilon : N \subset \Omega \}. \]

We have that
\[ |\Omega| - M_\Omega(\epsilon) \epsilon^m \geq 0. \]

In order to obtain an upper bound for the left hand-side of (8) we let \( x \in \Omega \). If \( d(x, \partial \Omega) > m^{1/2} \epsilon \), then \( x \) belongs to an open \( \epsilon \)-cube in \( \mathfrak{M}_\epsilon \) contained in \( \Omega \).
Hence the measure of the set which is not covered by the $\varepsilon$-cubes in $\mathcal{M}_\varepsilon$ that are entirely contained in $\Omega$ is bounded from above by $\mu_\Omega(m^{1/2}\varepsilon)$. So
\[
|\Omega| - M_\Omega(\varepsilon)\varepsilon^m \leq \mu_\Omega(m^{1/2}\varepsilon). \tag{9}
\]
By Dirichlet bracketing (see [15]) we have that
\[
N_\Omega(\lambda) \geq M_\Omega(\varepsilon)N_{C_\varepsilon}(\lambda), \tag{10}
\]
where $C_\varepsilon$ is an open cube in $\mathbb{R}^m$ with side-length $\varepsilon$. The following standard estimate is attributed to Gauss:
\[
N_{C_\varepsilon}(\lambda) = \{ (k_1, \ldots, k_m) \in \mathbb{N}^m : \sum_{i=1}^m k_i^2 < \pi^{-2}\varepsilon^2 \lambda \}
\geq \frac{\omega_m}{2\pi} m \left( \pi^{-1}\varepsilon^{1/2} - m^{1/2} \right) +
\geq \frac{\omega_m}{2\pi} \varepsilon^m \lambda^{m/2} \left( 1 - \frac{\pi m^{3/2}}{\varepsilon \lambda^{1/2}} \right), \tag{11}
\]
where $+$ denotes the positive part. By (10) and (11),
\[
N_\Omega(\lambda) \geq M_\Omega(\varepsilon)N_{C_\varepsilon}(\lambda)
\geq M_\Omega(\varepsilon) \frac{\omega_m}{(2\pi)^m} \varepsilon^m \lambda^{m/2} - M_\Omega(\varepsilon) \frac{\omega_m}{(2\pi)^m} \pi m^{3/2} \varepsilon^{-1} \lambda^{(m-1)/2}
= \frac{\omega_m}{(2\pi)^m} |\Omega| \lambda^{m/2} - (|\Omega| - M_\Omega(\varepsilon)\varepsilon^m) \frac{\omega_m}{(2\pi)^m} m^{3/2} \varepsilon^{-1} \lambda^{(m-1)/2}
- M_\Omega(\varepsilon) \frac{\omega_m}{(2\pi)^m} \pi m^{3/2} \varepsilon^{-1} \lambda^{(m-1)/2}. \tag{12}
\]
We bound the second and third terms in the right hand-side of (12) using (9) and (8) respectively. This then gives, by (7), that
\[
R_\Omega(\lambda) \leq \frac{\omega_m}{(2\pi)^m} \mu_\Omega(m^{1/2}\varepsilon) \lambda^{m/2} + \frac{\pi m^{3/2} \omega_m |\Omega| \lambda^{(m-1)/2}}{(2\pi)^m} \frac{\varepsilon}{\lambda}. \tag{13}
\]
By (6) and (13) we have that if $\lambda_\Omega(\Omega)$ is Courant-sharp then
\[
\frac{\omega_m}{(2\pi)^m} (1 - \gamma_m) |\Omega| (\lambda_\Omega(\Omega))^{m/2} \leq \frac{\omega_m}{(2\pi)^m} \mu_\Omega(m^{1/2}\varepsilon) (\lambda_\Omega(\Omega))^{m/2}
+ \frac{\pi m^{3/2} \omega_m |\Omega| (\lambda_\Omega(\Omega))^{(m-1)/2}}{(2\pi)^m} \frac{\varepsilon}{\lambda}. \tag{14}
\]
We now choose $\varepsilon$ such that the second term in the right hand-side of (14) equals half of the left hand-side of (14). That is
\[
\varepsilon = 2\pi m^{3/2} (1 - \gamma_m)^{-1} (\lambda_\Omega(\Omega))^{-1/2}. \tag{15}
\]
By (14) and the choice of $\varepsilon$ in (15) we have that if $\lambda_\Omega(\Omega)$ is Courant-sharp then
\[
2^{-1} (1 - \gamma_m) |\Omega| \leq \mu_\Omega(2\pi m^2 (1 - \gamma_m)^{-1} (\lambda_\Omega(\Omega))^{-1/2}). \tag{16}
\]
Since $\varepsilon \mapsto \mu_\Omega(\varepsilon)$ is continuous and onto $[0, |\Omega|]$ the infimum in (3) is over a non-empty set which is bounded from below, and therefore exists. So if $\lambda_\Omega(\Omega)$
is Courant-sharp then, by (3) and (16),
\[ \frac{2\pi m^2}{(1-\gamma_m)(\lambda_n(\Omega))^{1/2}} \geq \epsilon(\Omega). \]
This proves Theorem 1(i).

By [11] we also have that
\[ \lambda_n(\Omega) \geq \frac{m}{m+2} \frac{(2\pi)^2}{\omega_m^2} \left( \frac{n}{|\Omega|} \right)^{2/m}, \]
This, together with (4), implies (5) and proves Theorem 1(ii).

To prove Theorem 1(iii) we just note that by (17),
\[
\max \left\{ n \in \mathbb{N} : \lambda_n(\Omega) \leq \left( \frac{2\pi m^2}{(1-\gamma_m)\epsilon(\Omega)} \right)^2 \right\} \leq \frac{\omega_m}{(1-\gamma_m)m} \left( \frac{m^3(m+2)}{\epsilon(\Omega)m} \right)^{m/2} \frac{|\Omega|}{|\Omega|^{m-1}}. \]

We note that if we were to use the lower bounds for the counting function from Section 2 in [5] then we would have to assume a weak integrability condition on $\mu_1$ of the form $\int \epsilon^{-1} d\mu_1(\epsilon) < \infty$. Such an integrability condition may fail if the interior Minkowski dimension of $\partial \Omega$ is equal to $m$. The procedure above avoids this integrability condition.

3 Examples

In this section we analyse three examples where explicit computations seem out of reach.

Example 1. Let $\Omega$ be an open, bounded, convex set in $\mathbb{R}^m$. Let $\mathcal{H}^{m-1}(\partial \Omega)$ denote the $(m-1)$-dimensional Hausdorff measure of $\partial \Omega$. Then

\[ \mathcal{C}(\Omega) \leq \frac{\omega_m}{(1-\gamma_m)^{2m}} \left( \frac{4m^3(m+2)}{\epsilon(\Omega)} \right)^{m/2} \frac{\mathcal{H}^{m-1}(\partial \Omega)^m}{|\Omega|^{m-1}}. \]  

Proof. By convexity of $\Omega$ we have that
\[ \mu_1(\epsilon) \leq \mathcal{H}^{m-1}(\partial \Omega) \epsilon. \]

By (3),
\[ \epsilon(\Omega) \geq 2^{-1}(1-\gamma_m) \frac{|\Omega|}{\mathcal{H}^{m-1}(\partial \Omega)}, \]
and (18) follows from Theorem 1 and (19). \hfill \Box

It was shown in [10] that only the first, second and fourth Dirichlet eigenvalues for $\mathcal{B}_2$ are Courant-sharp. Hence $\mathcal{C}(\mathcal{B}_2) = 3$, and the largest Courant-sharp eigenvalue for $\mathcal{B}_2$ is equal to $j_{3,2}^2$. Here $j_{3,2} \approx 5.520$ is the second positive zero of the Bessel function $J_0$. A straightforward computation using (4) and (19) shows that the largest Courant-sharp eigenvalue of $\mathcal{B}_2$ is strictly less than $1.2 \cdot 10^6$. This compares well with the bound $7.1 \cdot 10^6$ obtained in [1]. For the unit square $\mathcal{C}_2$ it is known ([13], [2]) that only the first, second and fourth Dirichlet eigenvalues are Courant-sharp. Hence $\mathcal{C}(\mathcal{C}_2) = 3$, and the largest Courant-sharp eigenvalue for $\mathcal{C}_2$ is equal to $8\pi^2$. Using (4) and (19) we have that the largest Courant-sharp eigenvalue of the unit square is strictly less than $4.5 \cdot 10^6$, whereas
[1] gives a bound $5.9 \cdot 10^6$. These examples illustrate that the bounds obtained in Theorem 1 are very crude.

The second example is a von Koch snowflake $K$ with similarity ratio $\frac{1}{3}$. We recall its construction. Let the basic square (generation 0) in $K$ have side-length 1. The first generation consists of 4 squares with side-length $\frac{1}{3}$ each attached symmetrically to the basic square. Proceeding inductively we have that the $j$'th generation in $K$, $j \in \mathbb{N}$ consists of $4 \cdot 5^{j-1}$ squares with side-length $3^{-j}$. We let $K$ be the interior of its closure. Then $K$ is connected, has Lebesgue measure $|K| = 2$, and both the Hausdorff dimension of $\partial K$ and the interior Minkowski dimension of $\partial K$ are equal to $\log 5 / \log 3$. See Figure 1, and [4] for further details.

![Figure 1: The first two generations of $K$](image)

**Example 2.** Let $K$ be the von Koch snowflake generated by the unit square and similarity ratio $\frac{1}{3}$. Then

$$\mathcal{C}(K) \leq 15 \cdot 10^7.$$  \hspace{1cm} (20)

**Proof.** By Theorem 1, (2), and $|K| = 2$ we find that

$$\mathcal{C}(K) \leq \frac{64\pi j_0^2}{(j_0^2 - 4)^2} \epsilon(K)^{-2},$$  \hspace{1cm} (21)

where we have used that

$$\lambda_1(B_2) = j_0^2,$$

where $j_0 = 2.405...$ is the first positive zero of the Bessel function $J_0$. It remains to find a lower bound for $\epsilon(K)$. We obtain an upper bound for $\mu_1(\epsilon)$ by adding all edges between squares of different generations. This gives a disjoint union of 1 unit square and $4 \cdot 5^{j-1}$ squares with side-lengths $3^{-j}, j \in \mathbb{N}$. Let $\epsilon < \frac{1}{18}$. 

and let \( J \in \mathbb{N} \) be such that
\[
J < \frac{\log \left( \frac{1}{2} \right)}{\log 3} \leq J + 1.
\]
Then \( J \geq 2 \). The contribution to the upper bound for \( \mu_{\Omega}(\epsilon) \) from the squares in generations 1, \ldots, \( J - 1 \) is bounded from above by
\[
\left( 4 + 16 \sum_{j=1}^{J-1} 5^{j-1} 3^{-j} \right) \epsilon \leq \frac{24 \epsilon}{5} \left( \frac{5}{3} \right)^{J} \leq \frac{48}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon 2^{-\frac{\log 5}{\log 3}}. \quad (22)
\]
The first term in the left-hand side above is the contribution from the unit square. The contribution to the upper bound for \( \mu_{\Omega}(\epsilon) \) from the squares in generations \( J, J + 1, \ldots \) is bounded from above by
\[
\sum_{j \geq J} 4 \cdot 5^{j-1} 9^{-j} = \left( \frac{5}{9} \right)^{J-1} \leq \frac{36}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon 2^{-\frac{\log 5}{\log 3}}. \quad (23)
\]
We recognise the interior Minkowski dimension \( \frac{\log 5}{\log 3} \) of \( \partial K \). By (22) and (23) we have that
\[
\mu_{\Omega}(\epsilon) \leq \frac{84}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon 2^{-\frac{\log 5}{\log 3}} , \quad 0 < \epsilon < \frac{1}{18}.
\]
Solving the equation
\[
\frac{84}{5} 2^{-\frac{\log 5}{\log 3}} \epsilon 2^{-\frac{\log 5}{\log 3}} = 1 - \frac{4}{J_{0}}
\]
gives that
\[
\epsilon(K) \geq 0.00379. \quad (24)
\]
The bound of (20) follows by (21) and (24).

Below we construct an open set \( D_s \subset \mathbb{R}^3 \). Let \( Q_0 \subset \mathbb{R}^3 \) be an open cube of side-length 1. Let \( 0 < s \leq \sqrt{2} - 1 \). Attach a regular open cube \( Q_{1,i} \) of side-length \( s \) to the centre \( c_{1,i}, i = 1, \ldots, 6 \), of each face of \( \partial Q_0 \), and such that all the faces are pairwise-parallel. Now proceed by induction. For \( j = 2, 3, \ldots \), attach \( N(j) = 6 \cdot 5^{j-1} \) open cubes \( Q_{j,1}, \ldots, Q_{j,N(j)} \), of side-length \( s^j \) to the centres of the boundary faces of the cubes \( Q_{j-1,1}, \ldots, Q_{j-1,N(j-1)} \), again with pairwise-parallel faces. We define the polyhedron \( D_s \) as
\[
D_s = \text{interior} \left( Q_0 \cup \bigcup_{j \geq 1} \bigcup_{1 \leq i \leq N(j)} Q_{j,i} \right).
\]
See Figure 2. We note that for \( 0 < s \leq \sqrt{2} - 1 \) no cubes in the construction of \( D_s \) overlap.

The asymptotic behaviour of the heat content of \( D_s \) in \( \mathbb{R}^3 \) for small time was analysed in [6]. Here we have the following.

\textbf{Example 3.} Let \( s \in (0, \sqrt{2} - 1] \), and let \( D_s \) be the polyhedron in \( \mathbb{R}^3 \) defined above. Then
\[
\mathcal{C}(D_s) \leq 25 \cdot 10^{10}. \quad (25)
\]
Figure 2: The first two generations of $D_s$ with $s = \frac{1}{3}$.

Proof. We have that
\[ |D_s| = \frac{1 + s^3}{1 - 5s^2}, \]
and that the two-dimensional Hausdorff measure of the boundary is given by
\[ H^2(\partial D_s) = 6 \left( \frac{1 - s^2}{1 - 5s^2} \right). \]

By Theorem 1 we have that
\[ C(D_s) \leq \frac{36(15)^{3/2}}{\pi^2} \frac{|D_s|}{\epsilon(D_s)^3}, \quad (26) \]
where we have used that
\[ \lambda_1(B_3) = j_{1/2}^2 = \frac{\pi^2}{12}, \]
where $j_{1/2} = \pi$ is the first positive zero of the Bessel function $J_{1/2}$. We obtain an upper bound for $\mu_1(\epsilon)$ by adding all faces between cubes of different generations. This gives a disjoint union of 1 unit cube and $6 \cdot 5^{j-1}$ cubes of side-length $s^j$, $j \in \mathbb{N}$. Hence
\[ \mu_1(\epsilon) \leq \left( 6 + 36 \sum_{j=1}^{\infty} 5^{j-1} s^{2j} \right) \epsilon = \frac{6(1 + s^2)}{1 - 5s^2} \epsilon. \quad (27) \]

By (3) and (27) we have that
\[ \epsilon(D_s) \geq \frac{1}{12} \left( 1 - \frac{9}{2\pi^2} \right) \frac{1 - 5s^2}{1 + s^2} |D_s|. \quad (28) \]

Finally by (26), (28), the fact that $0 < s \leq \sqrt{2} - 1$, and $|D_s| \geq 1$ we obtain that
\[ \mathcal{C}(D_s) \leq 6(12)^4 (15)^{3/2} (140 + 99\sqrt{2}) \pi \left( 1 - \frac{9}{2\pi^2} \right)^{-6}. \]
This implies (25). \qed
References


