



Berry, M., & Morley-Short, S. (2017). Representing fractals by superoscillations. *Journal of Physics A: Mathematical and Theoretical*, 50(22), [LT01]. <https://doi.org/10.1088/1751-8121/aa6fba>

Peer reviewed version

Link to published version (if available):  
[10.1088/1751-8121/aa6fba](https://doi.org/10.1088/1751-8121/aa6fba)

[Link to publication record in Explore Bristol Research](#)  
PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via IOP Science at <http://iopscience.iop.org/article/10.1088/1751-8121/aa6fba/meta;jsessionid=1C2CD0CECC3BB50EBF2678E41A6DC0D1.c3.iopscience.cld.iop.org>. Please refer to any applicable terms of use of the publisher.

## **University of Bristol - Explore Bristol Research**

### **General rights**

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:  
<http://www.bristol.ac.uk/pure/about/ebr-terms>

# Representing fractals by superoscillations

M V Berry<sup>1</sup> & S Morley-Short<sup>2</sup>

<sup>1</sup>H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

<sup>2</sup>Quantum Engineering Technology Labs and Centre for Doctoral Training, H. H. Wills Physics Laboratory and Department of Electrical & Electronic Engineering, University of Bristol, Bristol BS8 1FD, UK

<sup>1</sup>asymptotico@bristol.ac.uk <sup>2</sup>sam.morley-short@bristol.ac.uk

## Abstract

Fractals provide an extreme test of representing fine detail in terms of band-limited functions, i.e. by superoscillations. We show that this is possible, using the example of the Weierstrass nondifferentiable fractal. If this is truncated at an arbitrarily fine scale, it can be expressed to any desired accuracy with a simple superoscillatory function. In illustrative simulations, fractals truncated with fastest frequency  $2^{16}$  are easily represented by superoscillations with fastest Fourier frequency 1.

**Keywords:** scaling, approximation, band-limited, Fourier

## 1. Introduction

Two mathematical-physics concepts that have been extensively studied in recent decades are fractals and superoscillations. Our purpose here is to show how they can be connected.

Fractals [1-3] are geometric structures with a hierarchy of scales extending infinitely small: they contain exact or statistically similar copies of themselves, and can be described by a noninteger dimension  $D$ ; for fractal curves in the plane (e.g. graphs of functions),  $1 < D < 2$ . Fractals have been applied to scale-invariant phenomena in many areas, including geophysics, biological branching (blood vessels, lungs, trees...), critical states in statistical mechanics, the border of chaos in dynamical systems, and signal and image processing. Superoscillations [4-8] are variations faster than the fastest Fourier components of band-limited functions. They have been applied to quantum weak measurements [9-11], speckle [12, 13], superresolution microscopy [14-20], radar superdirectivity [21, 22], and optical vortices [23, 24].

It is known that superoscillations can be adapted to represent not only trigonometric oscillations but also small features of arbitrary shape (for example narrow antenna radiation patterns [22] and narrow Gaussians [25, 26]). But a particularly challenging application is to use superoscillations to approximate fractals: describing functions with infinitely fine scales by functions that are band-limited. To demonstrate that this is possible, we employ an example, namely the celebrated Weierstrass fractal [1, 2]: a function that is continuous everywhere but differentiable nowhere.

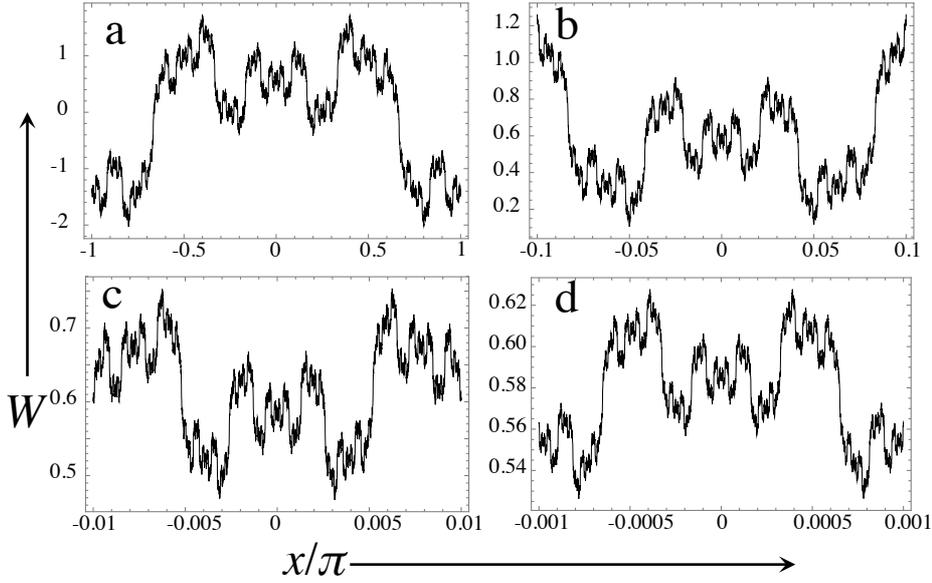
## 2. Theory

The Weierstrass fractal [1] can be conveniently written as

$$W(x, D, \gamma) = \sum_{n=0}^{\infty} \frac{\cos(\gamma^n x)}{\gamma^{n(2-D)}} \quad (\gamma > 1, 1 < D < 2), \quad (1)$$

in which  $\gamma$  is a multiplicative scaling factor describing levels of the hierarchy, and  $D$  is the fractal dimension of the graph of  $W$ ; for this function, the different definitions of fractal dimension [1] (for example Hausdorff-Besicovitch, Minkowski, and potential) coincide. If  $\gamma$  is integer,  $W(x, D, \gamma)$  is periodic with period  $2\pi$ . A slight modification, in which the sum is extended to negative integers  $n$ , and random phases are included in the sum, gives the Weierstrass-Mandelbrot function [3], which has no largest scale as well as no smallest scale, and enables the description of functions with statistical self-similarity; for present purposes, it is sufficient to consider simply  $W$ , defined by (1). For the reason why  $D$  is the fractal dimension, see [3, 27].

Truncating the sum in (1) at  $n=n_{\max}$  represents  $W$  down to scales  $\Delta x = \gamma^{-n_{\max}}$ . Figure 1 shows  $W$  over several intervals  $-X < x < X$ , illustrating how it scales under magnification.



**Figure 1.** The Weierstrass nondifferentiable function  $W(x)$  (equation 1) for fractal dimension  $D=3/2$  and scaling factor  $\gamma=2$ , showing successive magnifications over intervals  $-X \leq x \leq X$ , where (a)  $X=\pi$ , (b)  $X=0.1\pi$ , (c)  $X=0.01\pi$ , (d)  $X=0.001\pi$ , The sum in (1) is truncated at  $n=20$ , sufficient to show all scales that can be resolved visually

For our band-limited approximation to  $W$ , we will write the fast-varying factors  $\cos(\gamma^n x)$  in (1) in terms of the following well known superoscillatory function [9, 15]:

$$\begin{aligned}
 S(x, a, N) &= \text{Re} \left[ \left( \cos \frac{x}{N} + ia \sin \frac{x}{N} \right)^N \right] \\
 &= \left( \cos^2 \frac{x}{N} + a^2 \sin^2 \frac{x}{N} \right)^{N/2} \cos \left( N \tan^{-1} \left( a \tan \frac{x}{N} \right) \right) \quad (2) \\
 &\quad (a > 1, N \text{ even integer } \gg 1).
 \end{aligned}$$

This is periodic with period  $N\pi$ . It is band-limited because it can be written as a Fourier series with fastest component  $\cos x$ :

$$S(x, a, N) = \sum_{n=0}^{N/2} C_n \cos\left(\frac{2nx}{N}\right), \text{ where}$$

$$C_n = \frac{N!(-1)^{\frac{1}{2}N+n} (a^2 - 1)^{\frac{1}{2}N-n} \left( (a+1)^{2n} + (a-1)^{2n} \right)}{2^N (\frac{1}{2}N+n)! (\frac{1}{2}N-n)!} (1 - \frac{1}{2}\delta_{n,0}).$$

(3)

The superoscillatory behaviour of  $S$  for small  $x$  is exhibited by the expansion

$$S(x, a, N) = \cos(ax + O(x^3)) \exp\left(\frac{x^2}{2N}(a^2 - 1) + O(x^4)\right). \quad (4)$$

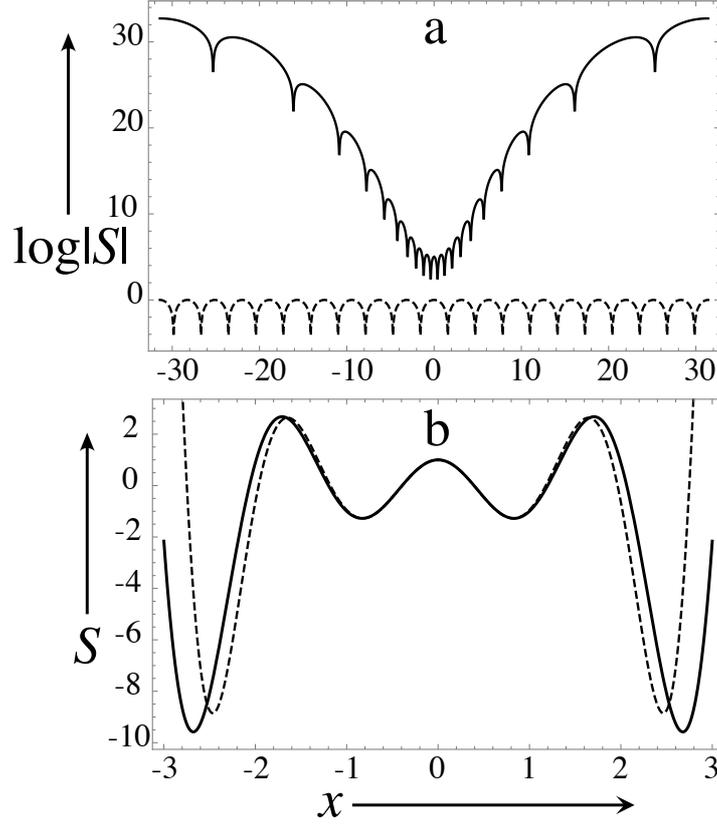
This shows that the parameter  $a$  describes the degree of superoscillatory compression:  $a$  times faster than the fastest Fourier component  $\cos x$ . The parameter  $N$  controls the extent of the region where superoscillation occurs, outside which  $S$  increases antigaussically. The superoscillatory behaviour and antigaussian increase are illustrated in figure 2. Explicitly, if we want to approximate  $S$  by  $\cos(ax)$  over the interval  $|x| < X$ , we must choose  $N$  sufficiently large:

$$S(x, a, N) \approx \cos(ax) \text{ if } |x| < X \text{ \& } N \gg N_{\min}(X, a), \quad (5)$$

where

$$N_{\min}(X, a) = \lceil X^2(a^2 - 1) \rceil \quad (6)$$

( $\lceil \dots \rceil$  denotes the (nearest integer above ...)).



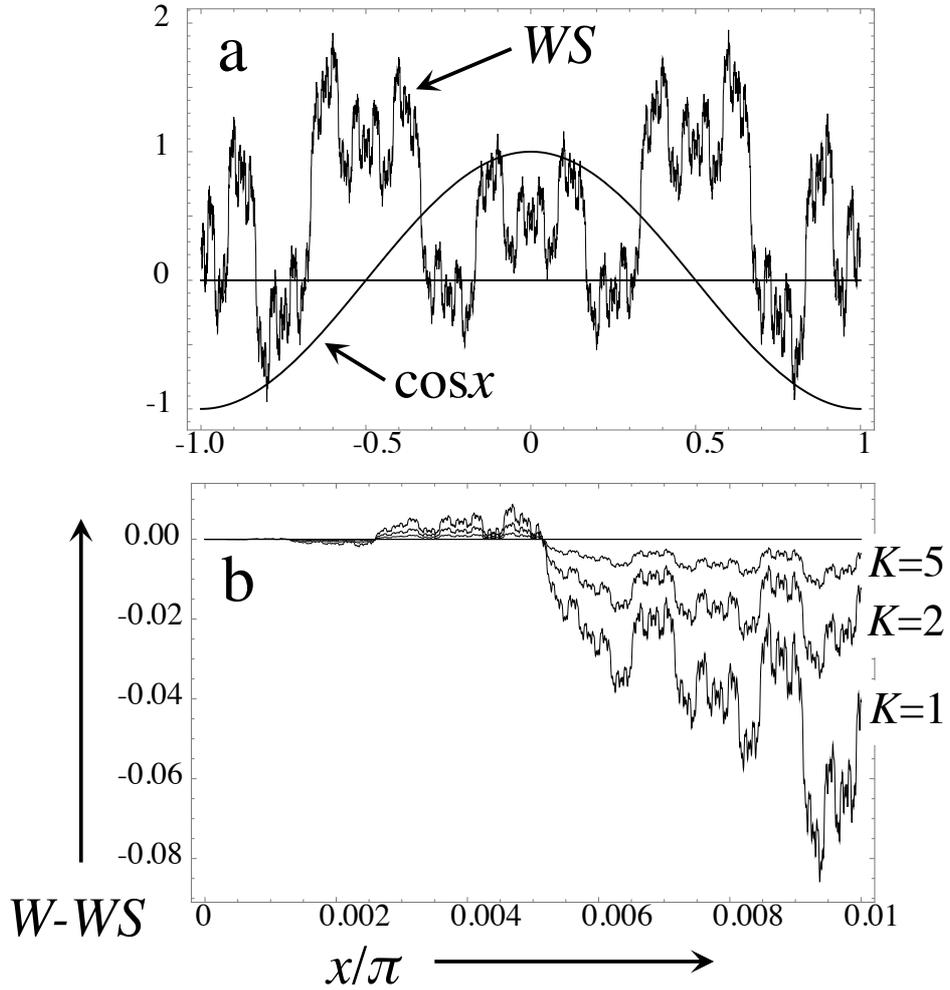
**Figure 2.** The superoscillatory function  $S(x)$  (equation 2) for  $a=4$ ,  $N=20$ . (a) full curve:  $\log|S(x)|$ ; dashed curve: fastest Fourier component  $\cos x$ , over a period  $-10\pi \leq x < 10\pi$ ; (b) full curve:  $S(x)$ ; dashed curve: antigaussian approximation (4), over the interval  $-3 \leq x < 3$ ; the approximation  $\cos(4x)$  is accurate over the interval  $-1.2 \leq x < 1.2$ , consistent with (6), which gives  $N_{\min}(1.2, 4)=22$ .

Now we can write the band-limited approximation of the Weierstrass fractal (1) over the interval  $-X < x < X$ , by replacing the fast-oscillating factors  $\cos(\gamma^n x)$  by their superoscillatory counterparts  $S$ , with  $a$  replaced by  $\gamma^n$ . To suppress the antigaussian increase of  $S$ , we choose  $N=KN_{\min}$  where  $K \gg 1$ . Thus, denoting the approximation by  $WS$ , we get our main result:

$$WS(x, D, \gamma; X, K) = \sum_{n=0}^{\infty} \frac{S(x, \gamma^n, KN_{\min}(X, \gamma^n))}{\gamma^{n(2-D)}}. \quad (7)$$

We expect the accuracy to increase with  $K$ ; however, figure 3a, which should be compared with figure 1a, illustrates that even with  $K=1$

the band-limited function  $WS$  can reproduce the fractal fine structure of the Weierstrass function. In figure 3a we have truncated the sum in (7) at  $n=16$ , so detail is reproduced down to the scale  $2^{-16}$ . Therefore the Weierstrass function (1), with fastest Fourier component  $\cos(2^{16}x)$ , is reproduced with a band-limited function whose fastest Fourier component is  $\cos x$  (also shown in figure 3a). The value of the parameter  $N$  corresponding to this truncation is  $N_{\min}=2^{32}-1=4294967295$ .



**Figure 3.** (a) The superoscillatory fractal function  $WS(x)$  (equation 7) for fractal dimension  $D=3/2$  and scaling factor  $\gamma=2$ , over the interval  $-X \leq x \leq X$  with  $X=\pi$  and antigaussian suppression factor  $K=1$ , with the sum truncated at  $n=16$ . Also shown is the fastest Fourier component  $\cos x$  in  $WS$ . (b) As (a) but showing the error  $W(x) - WS(x)$  in the approximation, for the indicated values of  $K$  and the smaller interval  $0 \leq x \leq 0.01\pi$ .

On casual visual inspection, the approximate figure 3a agrees with figure the exact figure 1a, but more careful examination indicates discrepancies increasing near the boundaries  $x=X=1$ . The approximation improves with increasing  $K$ ; to illustrate this, we show (figure 3b) the error  $W(x)-WS(x)$  for  $K=1, 2$  and  $5$ , over the more discriminating interval  $0 \leq x \leq 0.01\pi$ .

The approximation  $WS$  is not periodic, because the period  $\pi N$  of the component superoscillatory functions depends on  $n$ ; if the fractal sum is truncated at  $n$ , the period is  $\pi X^2(\gamma^{2n} - 1)$  – far larger than the range  $X$  being approximated.

As always superoscillations come at a cost.  $WS$  rises to enormous values outside the approximated range. From (2), (6) and (7), the maximum value of  $|S|$  is

$$|S_{\max}| = a^N = \gamma^{NK(X^2\gamma^{2n}-1)} . \quad (8)$$

For the parameters in figure 3a,  $|S_{\max}| \approx 10^{2.0 \times 10^{11}}$ . Associated with these large values is extreme sensitivity to noise [28].

### 3. Concluding remarks

The connection with fractals that we have explored here is perhaps the most stringent test of the ability of superoscillations to mimic fast-varying functions. Using the particular example of the Weierstrass function (1) and the simple canonical superoscillatory function (2), we have successfully demonstrated extreme frequency compression (e.g,  $2^{16}$  represented with fastest Fourier frequency 1 in figure 3).

It is obvious that the idea is more general. Other fractals could be represented band-limitedly; this is straightforward for fractals that are expressed as Fourier series, and a wider class of fractal patterns could be represented using superoscillatory functions of more than one variable [13, 29]. Examples of such patterns are optical waves in the Talbot effect from gratings with sharp edged rulings [30], and quantum waves evolving in enclosures from initial states with discontinuities [31, 32]. And although we have employed the superoscillatory function (2), the analysis could equally have been carried out with other elementary superoscillations (see e.g. [25]). Our analysis also suggests the possibility of further compression of images that have already been made smaller by fractal compression [33-36]. In practice, extreme compression might be compromised by noise, as already mentioned, because of the very large Fourier coefficients whose almost-complete cancellation is responsible for superoscillations [28]. We have not explored the intriguing possibility of using superoscillations to represent multifractals [37, 38], in which there is a spectrum of scaling exponents.

We have discussed the function  $WS$  as an approximation to the fractal  $W$ . In the limit  $K \rightarrow \infty$  in (7),  $WS$  would be a formally exact representation of  $W$ . The limit is very singular, because of the enormous increase of  $S$  outside the interval where it superoscillates, especially for the very large values  $N \gg N_{\min}$  that are relevant here. The precise mathematical sense in which  $WS$  represents  $W$  in the limit  $K \rightarrow \infty$  would be an interesting project, extending existing rigorous studies [5, 39] of related singular limits involving superoscillations.

**Acknowledgment.** We thank two referees, and Professor P Shukla, for helpful comments. MVB's research is supported by an Emeritus Fellowship from the Leverhulme Trust. SMS is supported by the Bristol Quantum Engineering Centre for Doctoral Training, EPSRC grant EP/L015730/1.

## References

1. Mandelbrot, B. B.,1982, *The Fractal Geometry of Nature* (Freeman, San Francisco)
2. Falconer, K.,1990, *Fractal Geometry: Mathematical Foundations and Applications* (Wiley, New York)
3. Berry, M. V. & Lewis, Z. V.,1980, On the Weierstrass-Mandelbrot fractal function *Proc.Roy.Soc.* **A370**, 459-484
4. Berry, M. V.,1994, *Faster than Fourier in Quantum Coherence and Reality; in celebration of the 60th Birthday of Yakir Aharonov* eds. Anandan, J. S. & Safko, J. L. (World Scientific, Singapore), pp. 55-65.
5. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D. & Tollaksen, J.,2017, The mathematics of superoscillations *Mem. Amer. Math. Soc.* **247**, no.1174
6. Kempf, A. & Ferreira, P. J. S. G.,2004, Unusual properties of superoscillating particles *J. Phys. A* **37**, 12067-12076

7. Ferreira, P. J. S. G. & Kempf, A.,2006, Superoscillations: faster than the Nyquist rate *IEEE Trans. Signal Processing* **54**, 3732-3740
8. Katzav, E. & Schwartz, M.,2013, Yield-Optimised Superoscillations *IEEE Trans. Signal Processing* **61**, 3113-3118
9. Aharonov, Y., Popescu, S. & Rohrlich, D.,1990, How can an infra-red photon behave as a gamma ray? *Tel-Aviv University Preprint TAUP*, 1847-1890
10. Aharonov, Y., Popescu, S. & Tollaksen, J.,2010, A time-symmetric formulation of quantum mechanics *Physics Today* **63**, issue 11, 27-33
11. Berry, M. V. & Shukla, P.,2012, Pointer supershifts and superoscillations in weak measurements *J.Phys.A.* **45**, 015301 (14pp)
12. Dennis, M. R., Hamilton, A. C. & Courtial, J.,2008, Superoscillation in speckle patterns *Optics Letters* **33**, 2976-2978
13. Berry, M. V. & Dennis, M. R.,2009, Natural superoscillations in monochromatic waves in  $D$  dimensions *J. Phys. A* **42**, 022003
14. Toraldo di Francia, G.,1952, SuperGain Antennas and Optical Resolving Power *Nuovo Cimento Suppl.* **9**, 426-438
15. Berry, M. V. & Popescu, S.,2006, Evolution of quantum superoscillations, and optical superresolution without evanescent waves *J.Phys.A.* **39**, 6965-6977

16. Rogers, E. T. F., Lindberg, J., Roy, T., Savo, S., Chad, J. E., Dennis, M. R. & Zheludev, N. I.,2012, A super-oscillatory lens optical microscope for subwavelength imaging *Nature Materials* **11**, 432-435
17. Leiserson, I., Lipson, S. G. & Sarafis, V.,2000, Superresolution in far-field imaging *Opt. Lett.* **25**, 209-211
18. Gazit, S., Szameit, A., Eldar, Y. & Segev, M.,2009, Super-resolution and reconstruction of sparse sub-wavelength images *Opt. Express* **17**, 23920-23946 (erratum **18**, 26631)
19. Shechtman, Y., Gazit, S., Szameit, A., Eldar, Y. & Segev, M.,2010, Super-resolution and reconstruction of sparse images carried by incoherent light *Opt. Lett.* **35**, 1148-1150
20. Berry, M. V.,2013, Exact nonparaxial transmission of subwavelength detail using superoscillations *J. Phys. A* **46**, 205203 (15pp)
21. Schelkunoff, S. A.,1943, A Mathematical Theory of Linear Arrays *Bell. Syst. Tech. J.* **22**, 80-107
22. Berry, M. V.,2013, *Superoscillations, endfire and supergain in Quantum Theory: a Two-time Success Story: Yakir Aharonov Festschrift* eds. Struppa, D. & Tollaksen, J. (Springer, New York), pp. 327-336.
23. Berry, M. V.,2008, *Waves near zeros in Coherence and Quantum Optics* eds. Bigelow, N. P., Eberly, J. H. & Stroud, C. R. J. (Optical Society of America, Washington DC), pp. 37-41.

24. Berry, M. V.,2013, A note on superoscillations associated with Bessel beams *J.Opt.* **15**, 044006 (5pp)
25. Berry, M. V.,2016, Representing superoscillations and narrow Gaussians by elementary functions *Milan J. Math.* **84**, 217-230
26. Ferreira, P. J. S. G. & Pinho, A. J.,2014, in International Conference on Acoustic, Speech and Signal Processing (ICASSP) (IEEE, pp. 4176-4179.
27. Orey, S.,1970, Gaussian Sample Functions and the Hausdorff Dimension of Level Crossings *Z. Wahrscheinlichkeitstheorie* **15**, 249-256
28. Berry, M. V.,2017, Suppression of superoscillations by noise *J. Phys. A* **50**, 025003 (9pp)
29. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D. & Tollaksen, J.,2016, Superoscillating sequences in several variables *J. Fourier. Anal. Appl.* **22**, 751-757
30. Berry, M. V. & Klein, S.,1996, Integer, fractional and fractal Talbot effects *J. Mod. Optics* **43**, 2139-2164
31. Berry, M. V.,1996, Quantum fractals in boxes *J. Phys. A* **26**, 6617-6629
32. Berry, M. V., Marzoli, I. & Schleich, W. P.,2001, Quantum Carpets, carpets of light *Physics World*, 39-44
33. Barnsley, M. F.,1988, *Fractals Everywhere* (Academic Press, Boston)

34. Barnsley, M. F.,1996, Fractal Image Compression *Not. AMS* **43**, 657-662
35. Saupe, D. & Hamzaoui, R.,1994, A Review of the Fractal Image Compression Literature *Computer Graphics* **28**, 268-276
36. Fisher, Y.,1995, *Fractal image compression* (Springer-Verlag, New York)
37. Stanley, H. E. & Meakin, P.,1988, Multifractal phenomena in physics and chemistry *Nature* **335**, 405-409
38. Harte, D.,2001, *Multifractals: theory and applications* (Chapman & Hall/CRC, London)
39. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D. C. & Tollaksen, J.,2013, On the Cauchy problem for the Schrödinger equation with superoscillatory initial data *J. Math. Pures. Appl.* **99**, 165-173

### Figure captions

**Figure 1.** The Weierstrass nondifferentiable function  $W(x)$  (equation 1) for fractal dimension  $D=3/2$  and scaling factor  $\gamma=2$ , showing successive magnifications over intervals  $-X \leq x \leq X$ , where (a)  $X=\pi$ , (b)  $X=0.1\pi$ , (c)  $X=0.01\pi$ , (d)  $X=0.001\pi$ , The sum in (1) is truncated at  $n=20$ , sufficient to show all scales that can be resolved visually

**Figure 2.** The superoscillatory function  $S(x)$  (equation 2) for  $a=4$ ,  $N=20$ . (a) full curve:  $\log|S(x)|$ ; dashed curve: fastest Fourier component  $\cos x$ , over a period  $-10\pi \leq x < 10\pi$ ; (b) full curve:  $S(x)$ ; dashed curve:

antigaussian approximation (4), over the interval  $-3 \leq x < 3$ ; the approximation  $\cos(4x)$  is accurate over the interval  $-1.2 \leq x < 1.2$ , consistent with (6), which gives  $N_{\min}(1.2, 4) = 22..$

**Figure 3.** (a) The superoscillatory fractal function  $WS(x)$  (equation 7) for fractal dimension  $D=3/2$  and scaling factor  $\gamma=2$ , over the interval  $-X \leq x \leq X$  with  $X=\pi$  and antigaussian suppression factor  $K=1$ , with the sum truncated at  $n=16$ . Also shown is the fastest Fourier component  $\cos x$  in  $WS$ . (b) As (a) but showing the error  $W(x) - WS(x)$  in the approximation, for the indicated values of  $K$  and the smaller interval  $0 \leq x \leq 0.01\pi$ .