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PARTITION REGULARITY OF A SYSTEM OF DE AND HINDMAN

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Received: 8/1/13, Revised: 3/25/14, Accepted: 4/13/14, Published: 7/10/14

Abstract
We prove that a certain matrix, which is not image partition regular over $\mathbb{R}$ near zero, is image partition regular over $\mathbb{N}$. This answers a question of De and Hindman.

1. Introduction

Let $A$ be an integer matrix with only finitely many non-zero entries in each row. We call $A$ a kernel partition regular (over $\mathbb{N}$) if, whenever $\mathbb{N}$ is finitely colored, the system of linear equations $Ax = 0$ has a monochromatic solution; that is, there is a vector $x$ with entries in $\mathbb{N}$ such that $Ax = 0$ and each entry of $x$ has the same color. We call $A$ an image partition regular (over $\mathbb{N}$) if, whenever $\mathbb{N}$ is finitely colored, there is a vector $x$ with entries in $\mathbb{N}$ such that each entry of $Ax$ is in $\mathbb{N}$ and has the same color. We also say that the system of equations $Ax = 0$ or the system of expressions $Ax$ is partition regular.

The finite partition regular systems of equations were characterised by Rado [4]. Let $A$ be an $m \times n$ matrix and let $c^{(1)}, \ldots, c^{(n)}$ be the columns of $A$. Then $A$ has the columns property if there is a partition $[n] = I_1 \cup I_2 \cup \cdots \cup I_t$ of the columns of $A$ such that $\sum_{i \in I_t} c^{(i)} = 0$, and, for each $s$,

$$\sum_{i \in I_s} c^{(i)} \in \langle c^{(j)} : j \in I_1 \cup \cdots \cup I_{s-1} \rangle,$$

where $\langle \cdot \rangle$ denotes (rational) linear span and $[n] = \{1, 2, \ldots, n\}$.

**Theorem 1 ([4])**. A finite matrix $A$ with integer coefficients is kernel partition regular if and only if it has the columns property.

The finite image partition regular systems were characterised by Hindman and Leader [3].
In the infinite case even examples of partition regular systems are hard to come by: see [1] for an overview of what is known. De and Hindman [2, Q3.12] asked whether the following matrix was image partition regular.

$$
\begin{pmatrix}
1 & & & \\
1 & 1 & & \\
2 & 1 & & \\
2 & & 1 & \\
4 & & 1 & \\
4 & 1 & & \\
4 & & 1 & \\
\vdots & & \vdots & \ddots
\end{pmatrix}
$$

where we have omitted zeroes to make the block structure of the matrix more apparent. De and Hindman’s matrix corresponds to the following system of linear expressions.

$$
\begin{align*}
x_{21} + x_{22} & \quad x_{21} + 2y & \quad y \\
x_{22} + 2y & \\
x_{41} + x_{42} + x_{43} + x_{44} & \quad x_{41} + 4y \\
x_{42} + 4y & \\
x_{43} + 4y & \\
x_{44} + 4y & \\
& \quad \vdots \\
x_{2^{n-1} + \cdots + 2^{n-1}} & \quad x_{2^{n-1} + 2^n y} \\
& \quad \vdots \\
x_{2^n} & \quad x_{2^n + 2^n y} \\
& \quad \vdots
\end{align*}
$$

A matrix $A$ is called image partition regular over $\mathbb{R}$ near zero if, for every $\delta > 0$, whenever $(-\delta, \delta)$ is finitely colored, there is a vector $x$ with entries in $\mathbb{R} \setminus \{0\}$ such that each entry of $Ax$ is in $(-\delta, \delta)$ and has the same color. De and Hindman sought a matrix that was image partition regular but not image partition regular over $\mathbb{R}$ near zero. It is easy to show that the above matrix is not image partition regular over $\mathbb{R}$ near zero, so showing that it is image partition regular would provide an example.
The main result of this paper is that De and Hindman’s matrix is image partition regular.

**Theorem 2.** For any sequence \((a_n)\) of integer coefficients, the system of expressions

\[
\begin{align*}
  x_{11} & \quad x_{11} + a_1 y & \quad y \\
  x_{21} + x_{22} & \quad x_{21} + a_2 y & \quad x_{22} + a_2 y \\
  x_{31} + x_{32} + x_{33} & \quad x_{31} + a_3 y & \quad x_{32} + a_3 y & \quad x_{33} + a_3 y \\
  & \quad & \quad \vdots
\end{align*}
\]

is partition regular.

Taking \(a_n = n\) implies that De and Hindman’s matrix is image partition regular.

Barber, Hindman and Leader [1] recently found a different matrix that is image partition regular but not image partition regular over \(\mathbb{R}\) near zero. Their argument proceeded via the following result on kernel partition regularity.

**Theorem 3 ([1]).** For any sequence \((a_n)\) of integer coefficients, the system of equations

\[
\begin{align*}
  x_{11} + a_1 y &= z_1 \\
  x_{21} + x_{22} + a_2 y &= z_2 \\
  & \quad \vdots \\
  x_{n1} + \cdots + x_{nn} + a_n y &= z_n \\
  & \quad \vdots
\end{align*}
\]

is partition regular.

In Section 2 we show that Theorem 2 can almost be deduced directly from Theorem 3. The problem we encounter motivates the proof of Theorem 2 that appears in Section 3.
2. A Near Miss

In this section we show that Theorem 2 can almost be deduced directly from Theorem 3.

Let \( \mathbb{N} \) be finitely colored. By Theorem 3 there is a monochromatic solution to the system of equations

\[
\begin{align*}
\tilde{x}_{11} - a_1 y &= z_1 \\
\tilde{x}_{21} + \tilde{x}_{22} - 2a_2 y &= z_2 \\
&\vdots \\
\tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - n a_n y &= z_n \\
&\vdots
\end{align*}
\]

For each \( n \) and \( i \), set \( x_{ni} = \tilde{x}_{ni} - a_n y \). Then

\[
x_{n1} + \cdots + x_{nn} = \tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - n a_n y = z_n,
\]

and

\[
x_{ni} + a_n y = \tilde{x}_{ni},
\]

so we have found a monochromatic image for System (1). The problem is that we have not ensured that the variables \( x_{ni} = \tilde{x}_{ni} - a_n y \) are positive. In Section 3 we look inside the proof of Theorem 3 to show that we can take (most of) the \( \tilde{x}_{ni} \) to be as large as we please.

3. Proof of Theorem 2

The proof of Theorem 3 used a density argument. The (upper) density of a set \( S \subseteq \mathbb{N} \) is

\[
d(S) = \limsup_{n \to \infty} \frac{|S \cap [n]|}{n}.
\]

The density of a set \( S \subseteq \mathbb{Z} \) is \( d(S \cap \mathbb{N}) \). We call \( S \) dense if \( d(S) > 0 \). We shall use three properties of density.

1. If \( A \subseteq B \), then \( d(A) \leq d(B) \).

2. Density is unaffected by translation and the addition or removal of finitely many elements.

3. Whenever \( \mathbb{N} \) is finitely colored, at least one of the color classes is dense.
We will also use the standard notation for sumsets and difference sets

\[ A + B = \{ a + b : a \in A, b \in B \} \]
\[ A - B = \{ a - b : a \in A, b \in B \} \]
\[ kA = \sum_{i=1}^{k} A \] 
\[ k \text{ times} \]

and write \( m \cdot S = \{ ms : s \in S \} \) for the set obtained from \( S \) under pointwise multiplication by \( m \).

We start with two lemmas from [1].

**Lemma 4 ([1]).** Let \( A \subseteq \mathbb{N} \) be dense. Then there is an \( m \) such that, for \( n \geq 2/d(A) \), \( nA - nA = m \cdot \mathbb{Z} \).

**Lemma 5 ([1]).** Let \( S \subseteq \mathbb{Z} \) be dense with \( 0 \in S \). Then there is an \( X \subseteq \mathbb{Z} \) such that, for \( n \geq 2/d(S) \), we have \( S - nS = X \).

The following consequence of Lemmas 4 and 5 is mostly implicit in [1]. The main new observation is that the result still holds if we insist that we use only large elements of \( A \). Write \( A_{>t} = \{ a \in A : a > t \} \).

**Lemma 6.** Let \( A \) be a dense subset of \( \mathbb{N} \) that meets every subgroup of \( \mathbb{Z} \), and let \( m \) be the least common multiple of \( 1, 2, \ldots, \left\lfloor 1/d(A) \right\rfloor \). Then, for \( n \geq 2/d(A) \) and any \( t \),

\[ A_{>t} - nA_{>t} \supseteq m \cdot \mathbb{Z} \]

**Proof.** First observe that, for any \( t \), \( d(A_{>t}) = d(A) \). Let \( n \geq 2/d(A) \), and let \( X = A_{>t} - nA_{>t} \). For any \( a \in A_{>t} \), we have by Lemma 5 that

\[ (A_{>t} - a) - n(A_{>t} - a) = (A_{>t} - a) - (n+1)(A_{>t} - a), \]

and so

\[ X = X - A_{>t} + a. \]

Since \( a \in A_{>t} \) was arbitrary it follows that \( X = X + A_{>t} - A_{>t} \), whence \( X = X + l(A_{>t} - A_{>t}) \) for all \( l \). By Lemma 4 there is an \( m_t \in \mathbb{Z} \) such that, for \( l \geq 2/d(A) \),

\[ l(A_{>t} - A_{>t}) = m_t \cdot \mathbb{Z} \] . Therefore, \( X = X + m_t \cdot \mathbb{Z} \) and \( X \) is a union of cosets of \( m_t \cdot \mathbb{Z} \).

Since \( A \) contains arbitrarily large multiples of \( m_t \), one of these cosets is \( m_t \cdot \mathbb{Z} \) itself.

Since \( lA_{>t} - lA_{>t} \) contains a translate of \( A_{>t} \),

\[ 1/m_t = d(m_t \cdot \mathbb{Z}) \geq d(A), \]

and \( m_t \leq 1/d(A) \). So \( m_t \) divides \( m \) and

\[ A_{>t} - nA_{>t} \supseteq m \cdot \mathbb{Z}. \]

\[ \square \]
Lemma 6 will allow us to find a monochromatic image for all but a finite part of System (1). The remaining finite part can be handled using Rado’s theorem, provided we take care to ensure that it gives us a solution inside a dense color class.

**Lemma 7 ([1])**. Let $\mathbb{N}$ be finitely colored. For any $l \in \mathbb{N}$, there is a $c \in \mathbb{N}$ such that $c \cdot [l]$ is disjoint from the non-dense color classes.

We can now show that System (1) is partition regular.

**Proof of Theorem 2.** Let $\mathbb{N}$ be $r$-colored. Suppose first that some color class does not meet every subgroup of $\mathbb{Z}$; say some class contains no multiple of $m$. Then $m \cdot \mathbb{N}$ is $(r - 1)$-colored by the remaining color classes, so by induction on $r$ we can find a monochromatic image. So we may assume that every color class meets every subgroup of $\mathbb{Z}$.

Let $d$ be the least density among the dense color classes, and let $m$ be the least common multiple of $1, 2, \ldots, \lfloor 1/d \rfloor$. Then for any dense color class $A$, any $t$ and $n \geq 2/d$,

$$A_{>t} - nA_{>t} \supseteq m \cdot \mathbb{Z}.$$  

Now let $N = \lfloor 2/d \rfloor - 1$. We will find a monochromatic image for the expressions containing only $y$ and $x_{ni}$ for $n \leq N$ using Rado’s theorem. Indeed, consider the following system of linear equations.

\[
\begin{align*}
u_1 &= x_{11} \\
u_2 &= x_{21} + x_{22} \\
u_3 &= x_{31} + x_{32} + x_{33} \\
&\vdots \\
u_N &= x_{N1} + \cdots + x_{NN} \\
v_1 &= x_{N1} + a_{1y} \\
v_2 &= x_{21} + a_{2y} \\
v_3 &= x_{32} + a_{2y} \\
&\vdots \\
v_N &= x_{NN} + a_{Ny} \tag{2}
\end{align*}
\]

The matrix corresponding to these equations has the form

\[
\begin{pmatrix}
B & -I
\end{pmatrix}
\]

where $B$ is a top-left corner of the matrix corresponding to the expressions of System (1) and $I$ is an appropriately sized identity matrix. It is easy to check that this matrix has the columns property, so by Rado’s theorem there is an $l$ such that, whenever a progression $c \cdot [l]$ is $r$-colored, it contains a monochromatic solution to the equations of System (2).
Apply Lemma 7 to get \( c \) with \( c \cdot [ml] \) disjoint from the non-dense color classes. Then \( mc \cdot [l] \subseteq c \cdot [ml] \) is also disjoint from the non-dense color classes, and by the choice of \( l \) there is a dense color class \( A \) such that \( A \cap (mc \cdot [l]) \) contains a solution to System (2). Since the \( u_n, v_{ni} \) and \( y \) are all in \( A \), \( y \) and the corresponding \( x_{ni} \) make the first part of System (1) monochromatic.

Now \( y \) is divisible by \( m \), so for \( n > N \) we have that

\[-na_ny \in A_{>a_ny} - nA_{>a_ny},\]

so there are \( \tilde{x}_{ni} \) and \( z_n \) in \( A_{>a_ny} \) such that

\[-na_ny = z_n - \tilde{x}_{n1} - \cdots - \tilde{x}_{nn}.\]

Set \( x_{ni} = \tilde{x}_{ni} - a_ny \). Then

\[x_{n1} + \cdots + x_{nn} = \tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - na_ny = z_n,\]

and

\[x_{ni} + a_ny = \tilde{x}_{ni},\]

for each \( n > N \) and \( 1 \leq i \leq n \). Since \( \tilde{x}_{ni} \) and \( z_n \) are in \( A \) it follows that the whole of System (1) is monochromatic.

It remains only to check that all of the variables are positive. But for \( y \) and \( x_{ni} \) with \( n \leq N \) this is guaranteed by Rado’s theorem; for \( n > N \) it holds because \( \tilde{x}_{ni} > a_ny \).

\[\square\]

References


