Visualizing the structure of chaos in the Lorenz system

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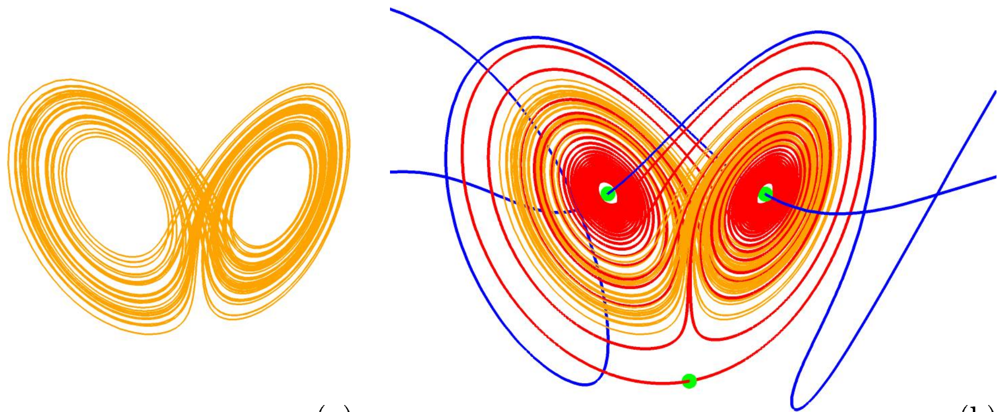
Abstract

The Lorenz attractor, with its characteristic butterfly shape, has become a much published symbol of chaos. It can be computed by simply integrating (virtually) any initial point. However, it is much more difficult to understand how the Lorenz attractor organizes the dynamics in a global way. We use a recently developed algorithm to compute a complicated two-dimensional surface called the stable manifold. Visualization tools are key to conveying its intricate geometry and beauty, which in turn provides insight into the structure of chaos in the Lorenz system.

Keywords: Lorenz system, chaos, strange attractor, invariant manifolds.

Note that the quality of the images in this preprint has been reduced substantially so that the electronic file has a reasonable size.



(a) (b)

Figure 1: *The Lorenz attractor obtained by integration (a). The attractor with the one-dimensional stable manifolds of the nonzero equilibria in blue and the one-dimensional unstable manifold of the origin in red (b).*

1 Introduction

The development of the theory of dynamical systems has benefited much from the visualization of chaotic attractors and other fractal objects. Computer experiments are useful not only to illustrate known properties of chaotic systems, but also to gain new mathematical insight. From a more artistic point of view, spectacular images of chaotic attractors and fractal sets have made their mark in popular culture and can now be found on anything from T-shirts to coffee mugs.

Chaotic attractors combine the regular with the irregular in an intriguing way. On the one hand, they attract all points nearby, so that starting at (virtually) any initial point always leads to a picture of the same object. On the other hand, if one follows two initial points that are arbitrarily close together on the attractor, they will move over the attractor in completely different ways after only a short period of time.

What does an attractor look like with these seemingly contradictory properties? The answer is 'strange', which is why chaotic attractors are also called strange attractors. Their strangeness makes chaotic attractors apparently intriguing and beautiful.

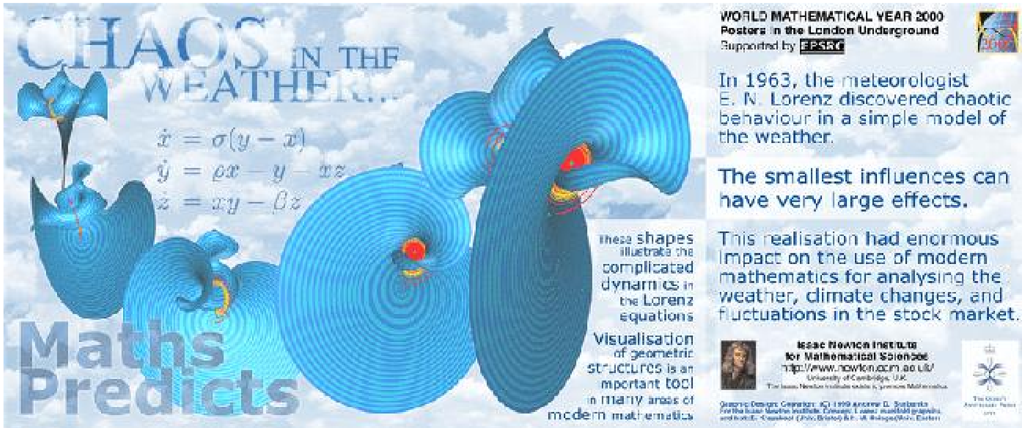


Figure 2: *The poster 'Maths Predicts' is part of the series 'Posters in the London Underground' celebrating the World Mathematics Year 2000. It was designed by Andrew D. Burbanks (University of Bristol) in collaboration with the authors. The series was organised by the Isaac Newton Institute for Mathematical Sciences with support from the UK's Engineering and Physical Sciences Research Council.*

While there is a large choice of chaotic attractors, we concentrate here on a classic one: the Lorenz attractor [4] shown in Figure 1(a); it was obtained by integrating an initial condition for 5000 integration steps. This chaotic attractor attracts practically all points in its three-dimensional phase space. Any point on the attractor will spiral around the center of one of the butterfly's wings for a while before shooting over to the other side and spiraling around the center of the other wing and so on. How long a point spirals and when it will shoot over to the other side depends so sensitively on the exact location of the initial point that it is unpredictable after only a short period of time. Figure 1(a) conveys the chaotic nature of the dynamics *on* the Lorenz attractor quite well, but does not give any insight into the overall dynamics of *the rest of the phase space.*

In this paper, we show how the Lorenz attractor organizes the dynamics in the whole of the phase space, which is mathematically interesting and visually appealing at the same time. Specifically, we compute the stable manifold of the origin, a complicated two-dimensional surface. Roughly speaking, it devides the phase space into points that first go to one wing of the butterfly and those that first go to the other wing when approaching the attractor; for more details see Section 2. The geometry of this surface is very intricate and the key to understanding the global chaotic dynamics. Using the visualization package Geomview [9], this surface is shown in different ways in order to convey its complexity and beauty. A first impression is given in Figure 2, which is a reproduction of a poster that was displayed in the London Underground in March 2000 as part of a series celebrating the World Mathematics Year 2000. It shows several images of the stable manifold as it starts to spiral into the Lorenz attractor.

This paper is organized as follows. In the next section we introduce the Lorenz system and the stable manifold of the origin. Section 3 describes the geometry of this manifold by using different ways to visualize it. In Section 4 we draw some conclusions, and in Appendix A we give technical details on how the graphics were rendered.

2 The Lorenz system

In 1963 the meteoroligist E.N. Lorenz [4] derived a much simplified 'weather model' (in fact, describing the dynamics of two scrolls transporting hot air up and cold air down in the atmosphere). Now called the Lorenz equations, this model is given by a set of three ordinary differential equations:

$$\left\{ \begin{array}{rcl} \dot{x} & = & \sigma(y-x), \\ \dot{y} & = & \varrho x - y - xz, \\ \dot{z} & = & xy - \beta z. \end{array} \right. \tag{1}$$

The classic choice of parameters is $\sigma = 10$, $\varrho = 28$, and $\beta = 2\frac{2}{3}$. Lorenz found the extreme sensitivity on initial conditions by accident: he restarted a computer calculation by copying the coordinates of an initial point from an earlier print-out. The round-off error on the print-out led to completely different results. He then realized that there was an underlying structure to this phenomenon and gave the first description of the chaotic attractor of Eqs. (1), which is now called the Lorenz attractor. For a mathematical derivation of the Lorenz equations see, for example, [8], and for an historic account see [2].

Understanding the exact nature of chaos in the Lorenz equations has been the motivation for a massive mathematical research effort. In fact, it was only proved in 2000 by Tucker [11] (with the help of rigorous computer estimates) that the Lorenz attractor is indeed a chaotic attractor! We refer the interested reader to the recent survey article by Viana [12] for more mathematical details.

We now introduce some technical terms from dynamical systems; see, for example, [7, 8, 10] as general background reading. The Lorenz system (1) is symmetric under the transformation $(x, y) \mapsto (-x, -y)$, which is the rotation about the z-axis over 180 degrees. It has three stationary points or equilibria: the origin and the two nonzero equilibria with $x = y = \pm\sqrt{\beta(\varrho - 1)}$ and $z = \varrho - 1$, which are images of each other under the symmetry. All equilibria are saddle points, meaning that they have both attracting and repelling directions. The origin is, in fact, part of the Lorenz attractor, and it has one unstable direction and two stable ones.

According to the Unstable Manifold Theorem we have the following. There is the one-dimensional unstable manifold, denoted $W^u(0)$, consisting of all points that end up at the origin when integrating backward in time. The unstable manifold $W^u(0)$ is a one-dimensional smooth curve tangent to the eigenspace of the unique unstable eigenvalue of the linearization of Eqs. (1) at the origin. One-dimensional stable and unstable manifolds can easily be computed by integrating a point in the stable or unstable eigenspace that lies close to the equilibrium (backward for stable and forward for unstable). The unstable manifold $W^u(0)$ accumulates on the chaotic attractor, as is shown in Figure 1(b). In fact, the origin is part of the attractor and $W^u(0)$ gives a better idea of the size of the attractor than only the orbit in Figure 1(a). Also shown in Figure 1(b) are the one-dimensional stable manifolds of the two nonzero equilibria, consisting of all points that end up at the respective equilibrium as time goes to plus infinity.

What we are concerned with here is the two-dimensional stable manifold of the origin, denoted $W^s(0)$, consisting of all points that end up at the origin as time goes to plus infinity. The stable manifold $W^s(0)$ is a two-dimensional smooth surface tangent to the eigenspace of the two stable eigenvalues at the origin. This manifolds is infinitely large!

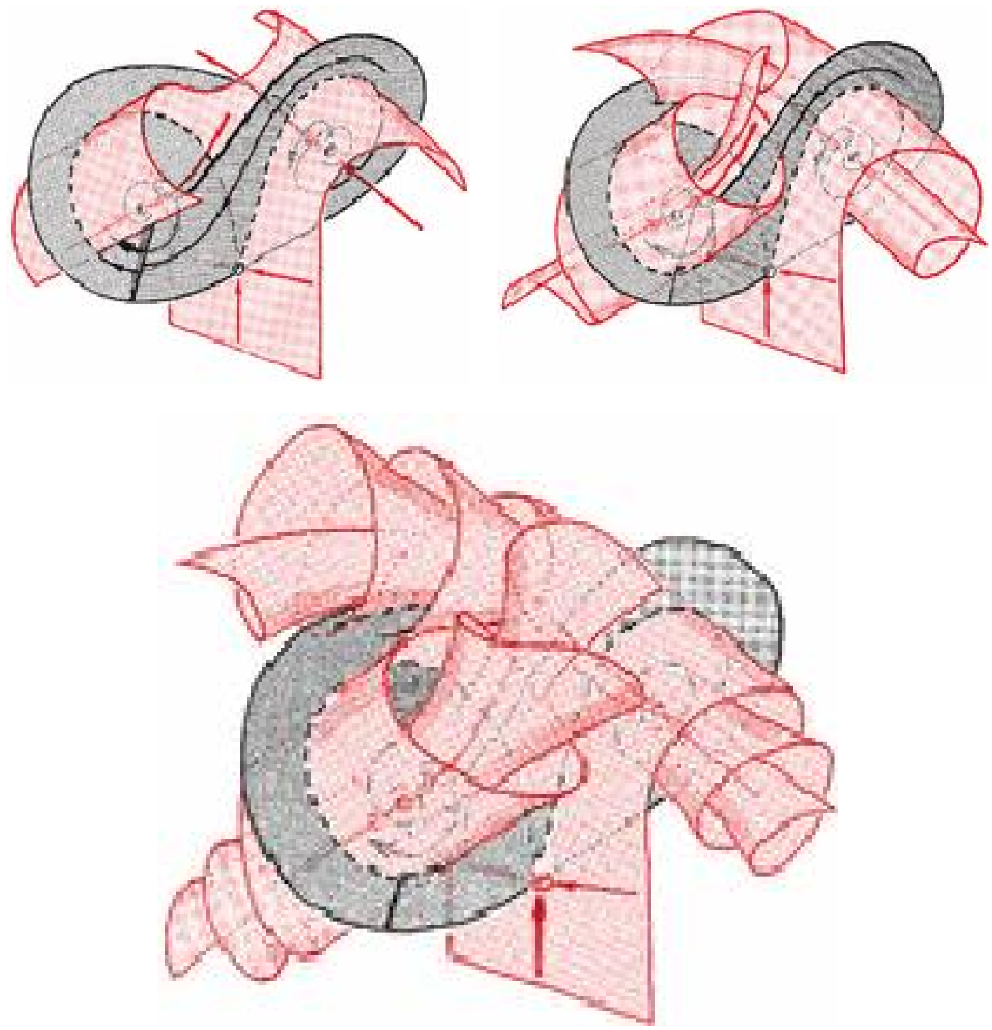


Figure 3: *Growth of the stable manifold (red surface) of the origin of the Lorenz system as sketched by Abraham and Shaw in 1982 [AS82]; reproduced with permission from Ralph Abraham and Aerial Press, Santa Cruz, CA.*

The big question is: what does the two-dimensional stable manifold $W^s(0)$ look like further away from the origin? A qualitative picture was given by Abraham and Shaw in 1982 [1], and this is reproduced here in Figure 3. By using only mathematical arguments they concluded that $W^s(0)$ must spiral into the Lorenz attractor as shown in the three consecutive sketches. Here, they used the fact that $W^s(0)$ must spiral around the one-dimensional stable manifolds of the two nonzero equilibria. These manifolds are sketched as almost straight lines in Figure 3. Our computations in Figure 1(b) show that the one-dimensional stable manifolds are, in fact, spiraling as well. The drawings in Figure 3 show a qualitative picture of the stable manifold, but what does $W^s(0)$ really look like?

It turns out that computing a stable or unstable manifold of dimension larger than one is such a difficult task that this question remained unanswered for a long time. However, quite recently algorithms were developed that allow one to compute such manifolds. The images of $W^s(0)$ in this paper were computed with the method we developed in [5, 6]. In particular, see [5] for general background information, more details and the algorithm in pseudo code. We started with a small circle formed by 20 points in the stable eigenspace $E^s(0)$. The algorithm builds up the manifold uniformly in all directions by growing level sets of points that lie at the same (geodesic) distance along $W^s(0)$ from the origin. Note that the geodesic level sets on $W^s(0)$ are smooth closed curves (topological circles). A new point in a new level set is found by solving an appropriate boundary value problem. In the course of the computation points are added or removed adaptively according to prespecified accuracy parameters in such a way that the mesh quality is guaranteed. The output is a set of *ribbons* formed by linear interpolation between two consecutively computed level sets. In total 72 circles were computed on $W^s(0)$; the last circle is approximately at geodesic distance 151.75 from the origin, and it consists of 2330 mesh points.

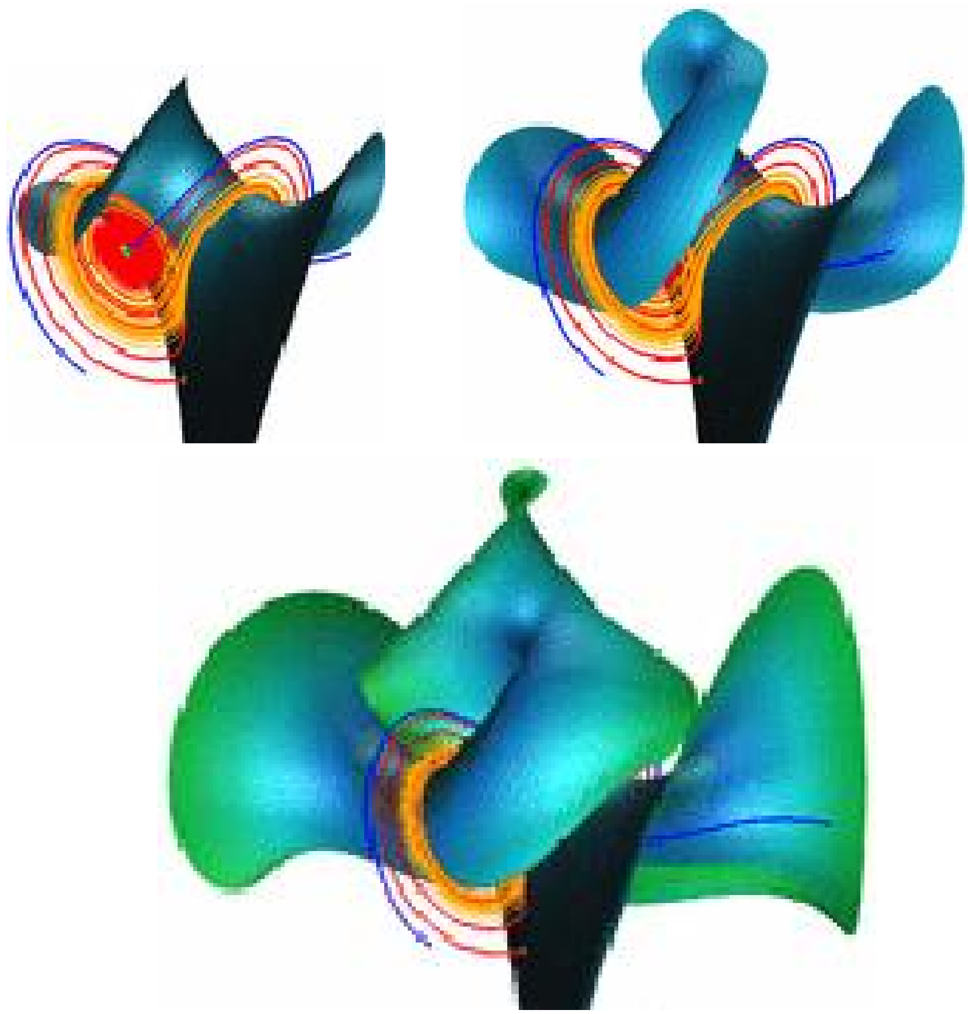


Figure 4: *Growth of the stable manifold of the origin of the Lorenz system computed with our manifold algorithm; compare Figure 3.*

3 Visualizing the stable manifold

Figure 4 shows our version of the sketches by Abraham and Shaw in Figure 3. We tried to immitate the viewpoint and how much of the manifold is shown. Apart from a cut-off at the negative z-axis, we show the *entire* manifold up to a specific geodesic distance. Throughout this paper, we show (parts of) our computed approximation of $W^s(0)$ as a surface that is colored according to how far (in geodesic distance) the points lie from the origin. Parts of the manifold that are blue are close to the origin, and parts that are red are far away; the precise color map is shown at the bottom of Figure 5.

Let us compare the topological sketches of Figure 3 with the numerical approximation in Figure 4. Note that Figure 3 does not show $W^s(0)$ up to a certain geodesic distance, such that some features are more apparent in this figure than in Figure 4 and vice versa. An immediate observation is that the spiraling behavior indicated in the drawing of Figure 3 does not seem to happen in the computed version. In fact, the computed manifold does spiral around the one-dimensional unstable manifold, but this is clear only when it is computed further; we discuss this more precisely in Section 3. Another striking difference is the absence of the part of the manifold that winds around the z-axis in Figure 3. However, this helix-like part of the manifold can be inferred from the sketches. The first numerical approximation of $W^s(0)$ appeared in 1993 [3, Figure 2 on p. 245] and already shows the helical structure around the z-axis.

The entire manifold up to geodesic distance 151.75 is shown from two different viewpoints in Figure 5. Recall that points on $W^s(0)$ are colored according to geodesic distance to the origin. (In contrast, on the poster in Figure 2 the ribbons are colored in alternating shades of blue.) Since $W^s(0)$ is a two-dimensional manifold, the geodesic level sets, formed by points that have the same color, are closed curves (topological circles). To give a better idea of how $W^s(0)$ is foliated with ribbons we plotted only every other ribbon in the pictures on the right of Figure 5. The viewpoints are identical to those on the left. A nice side-effect of visualizing only every second ribbon is that we can no "look inside" the manifold.

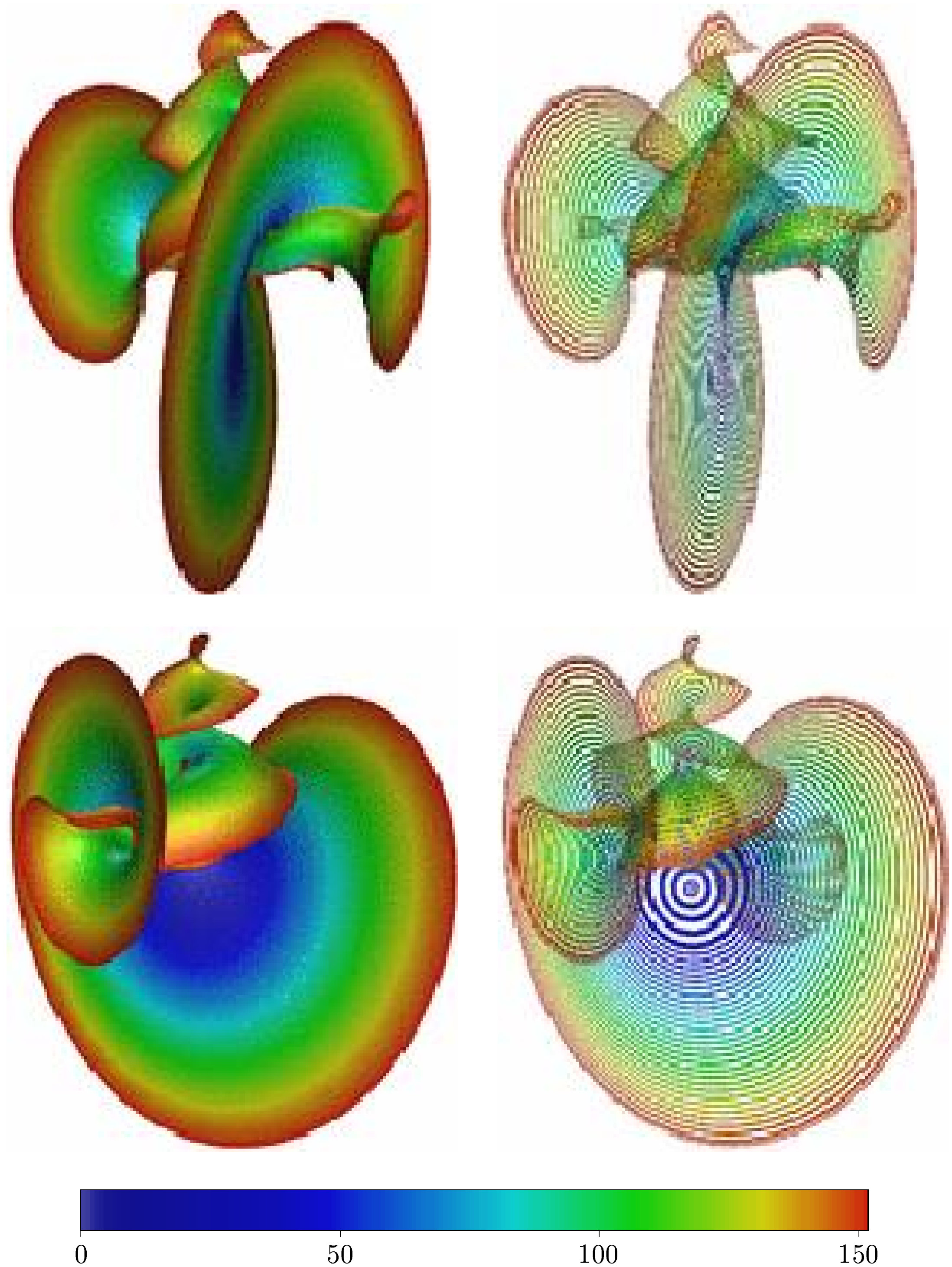


Figure 5: *Two views of the stable manifold of the origin, rendered as the entire surface (left) and as every second ribbon (right). The geodesic distance of points on the manifold from the origin is given by the color map at the bottom in this and all similar figures.*

The reader is invited to try to follow one of the ribbons in Figure 5. Particularly in the bottom right picture, we can see that the first (blue) level sets are almost perfectly planar circles. Further away from the origin these circles get distorted, as the manifold spirals into the attractor and simultaneously winds around the z-axis; compare also the top left picture in Figure 4. Even the ribbons at large distances (in red) are still unknotted topological circles!

There are two other interesting observations in Figure 5. First, the pictures clearly show how $W^s(0)$ spirals around the stable manifolds of the nonzero equilibria. Second, in the bottom left picture of Figure 5, one can spot a small red helix popping up right next to the larger helix around the z-axis. Due to the symmetry in the system, there are actually two of these, and they are both just visible in the picture on the bottom right. This is an interesting new observation, because up to now it was not known that $W^s(0)$ contains other helixes. They are a direct consequence of the fact $W^s(0)$ is foliated by smooth closed curves that are symmetrical with respect to the two "wings" of the butterfly attractor. Even though the computations were not carried out for larger geodesic distances, we conclude that further away from the origin even more helixes will appear in the vicinity of the z-axis.

The sketches in Figure 3 show that the two-dimensional stable manifold of the origin forms two spirals around the one-dimensional stable manifolds of the two nonzero equilibria as it grows further and further away from the origin. Even though this is not drawn explicitly, Figure 3 also indicates that each of these spirals of $W^s(0)$ must somehow also enter into the other spiral. This is indeed the case, as is shown in Figures 6 and 7, where the manifold $W^s(0)$ is clipped with solid spheres that are all centered at $(27, 0, 0)$ and have radii 35 (a), 50 (b), and 60 (c), respectively. The two figures are identical, except that the entire manifold is shown in Figure 6, while every other ribbon of the manifold is displayed in Figure 7. Each picture also shows the attractor, the three equilibria and (parts of) their one-dimensional stable or unstable manifolds. The attractor and the manifolds are shown from two angles, which are the same for (a)-(c), and we zoom out as the radius increases.

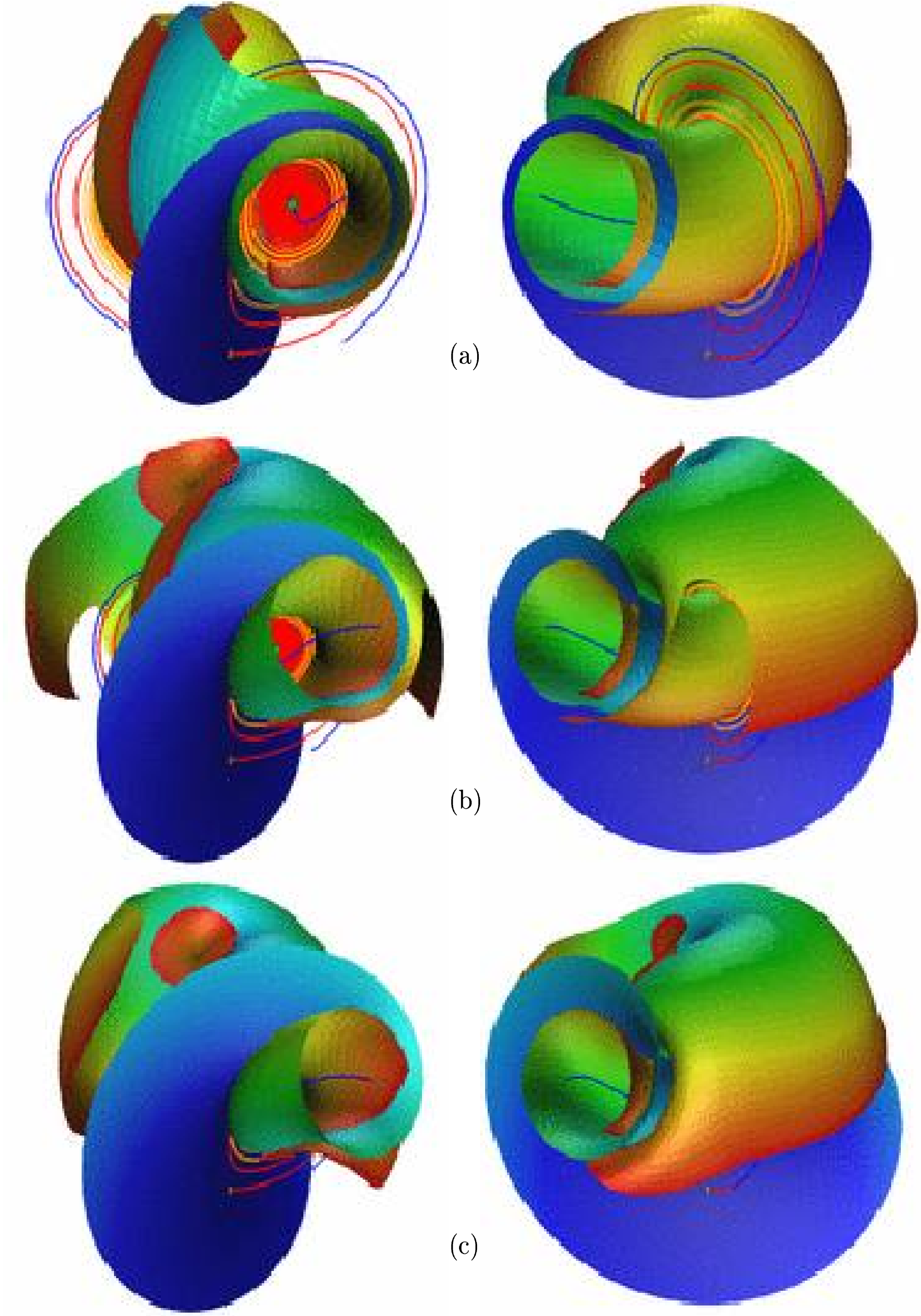


Figure 6: *Intersections of the entire computed manifold with solid spheres of radii 35 (a), 50 (b), and 60 (c) around (0,0,27) of the stable manifold of the origin.*

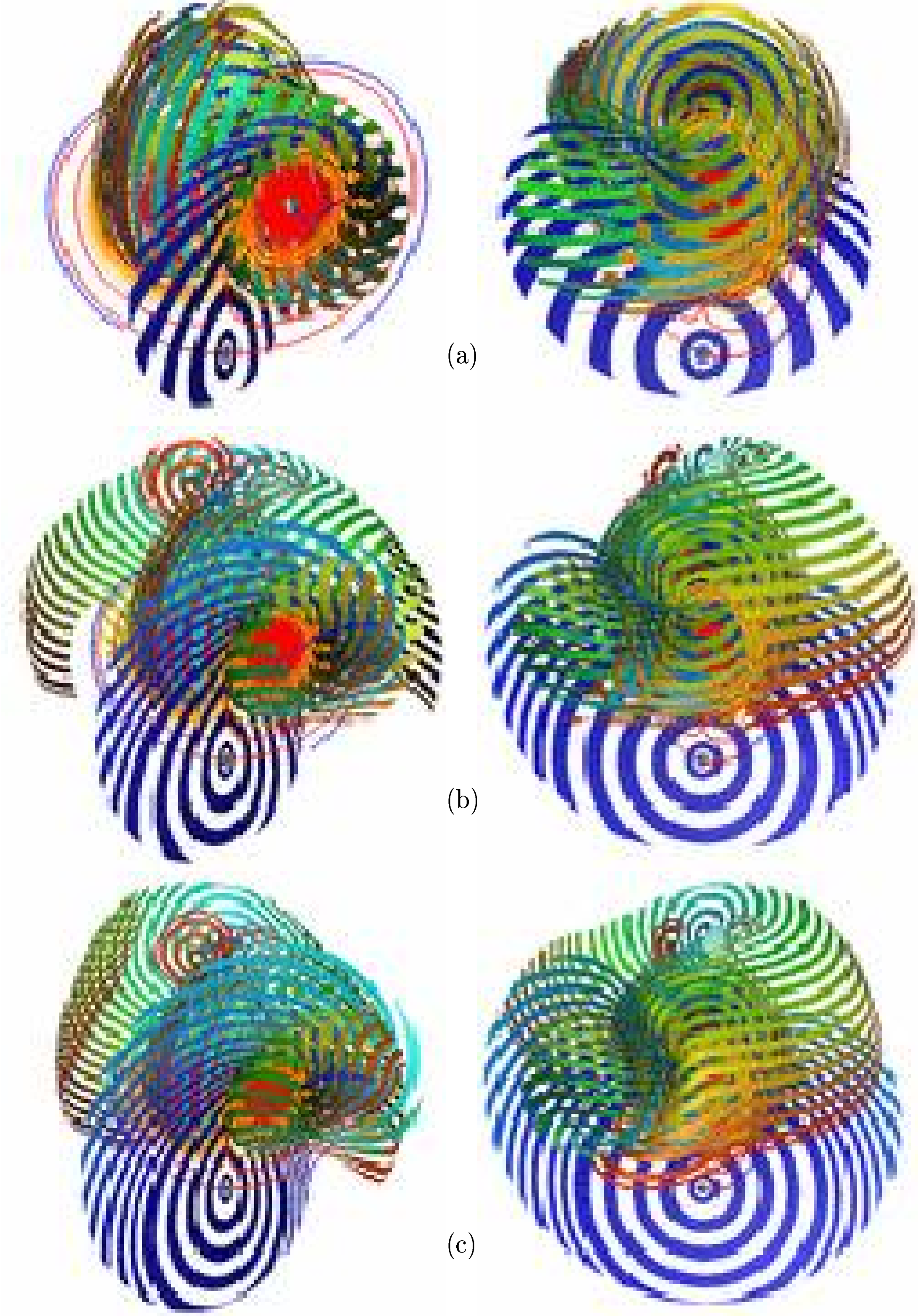


Figure 7: *Intersections of every other ribbon of the manifold with solid spheres of radii 35 (a), 50 (b), and 60 (c) around (0,0,27); the views are as in Fig. 6.*

We now discuss in more detail the structure of $W^s(0)$ revealed by the clipped renderings in Figures 6 and 7. Here the coloring scheme associated to geodesic distance is particularily helpful. In panels (a) one clearly sees that the stable manifold was computed until it makes two full turns around the one-dimensional stable manifold. Interestingly, a second sheet of the surface just starts to enter into this spiral. As can be seen from the view on the right, this sheet comes from the other side of the attractor, where it is part of the other spiral. The structure of a new spiral interleaving with the initial one is brought out clearly in the successive clippings in panels (b) and (c). Since points with the same color lie on a closed curve, the two spirals colored green-yellow-red in Figure 6 must somehow connect. The connection happens via the extra two helixes that were already pointed out in the discussion of Figure 5. This can be seen in the left of Figure 7(b): the symmetrical part of the spiral coming out of one wing of the butterfly attractor goes into the other wing. In the process the sheet forming the new spiral folds back over itself, as is clearly shown on the right of Figure 6(c).

In summary, the complex geometry of $W^s(0)$ discussed here can be explained by the fact that $W^s(0)$ goes through the Lorenz attractor without ever intersecting any trajectory on it. A typical trajectory on the attractor visits both the left and the right wings of the butterfly, while spending an arbitrarily long time in each wing. Even though the *geometry* of $W^s(0)$ is no indicator of the *dynamics* on $W^s(0)$, Figures 6(c) and 7(c) show that the structure of the manifold itself also involves visiting both wings of the butterfly attractor.

4 Conclusion

Algorithms are now available to compute two-dimensional stable or unstable manifolds of vector fields. These manifolds are complicated surfaces that reveal much about the dynamics of the system under consideration and they can only be understood by employing the latest visualization techniques. Our algorithm for computing manifolds is useful for a wealth of other systems; see the examples in [5, 6]. It is presently implemented for two-dimensional manifold in phase spaces of arbitrary dimension n. The visualization of such a manifold is a challenge for $n \geq 4$ because of artificial self-intersections induced by the necessary projections; an example of a manifold in a four-dimenional space can be found in [6].

In this paper we presented a study of the stable manifold $W^s(0)$ of the origin of the Lorenz system. The different ways of rendering this surface brought out the geometry as well as the beauty of this surface. Particularily helpful was the idea to color the manifold according to geodesic distance from the origin. The manifold $W^s(0)$ organises the dynamics of the Lorenz system. Trajectories starting on one side of this manifold cannot get to the other side: $W^s(0)$ is a *separating* manifold. It devides the phase space into points that first go to one wing of the butterfly and those that first go to the other wing when approaching the attractor. More generally, all trajectories, except those on $W^s(0)$ itself, are sandwiched between the sheets of $W^s(0)$. Knowing the whole of this manifold, therefore, would tell one the complicated dyamics of any point in phase space. Even though we only computed a finite piece of $W^s(0)$, this already hinted at how the strangeness of the butterfly attractor has a global effect and how its chaotic dynamics is felt infinitely far away from the attractor itself.

Acknowledgments

We would like to thank Mike Henderson for pointing out the sketches by Ralph Abraham and Christopher Shaw, and Aerial Press, Santa Cruz, CA, for giving us permission to reproduce Figure 3. We also thank Andy Burbanks who allowed us to reproduce the poster in Figure 2. We are grateful for the hospitality of the Control and Dynamical Systems Department at the California Institute of Technology in Pasadena, where part of this work was conducted.

A Technical information about the graphics

All figures in this paper were rendered with the visualization package Geomview [9], which is built on the Object Oriented Graphics Library (OOGL). Two consecutive circles computed by our algorithm are written to the data file in the Object File Format (OFF) such that a ribbon is formed with the two circles as its boundary. We refer to the Geomview online manual at `http://www.geomview.org/docs/html` for detailed information on Geomview and the used file format.

Geomview comes with a PERL script to clip the manifold against a plane, sphere, or cylinder. For example, the manifold displayed in Figure 6(a), is obtained by the command

```
clip -l 35 -sph 0,0,27 lorenz.off
```

where `lorenz.off` contains the original data of $W^s(0)$. A PERL script is also used to render only every second ribbon of the manifold.

We find that the standard options provided by Geomview for saving graphics are not sufficient for very high quality images. Instead, we use the following `gcl` command

```
(snapshot focus "| pnmscale .5 |
                   pnmtotiff > figure.tiff" ppm 1800)
```

which is typed in the "Command" window (under the "Inspect" menu). In this way, Geomview produces a 900×900 pixel picture in `tiff` format by making a snapshot of the "Camera" window at 1800×1800 pixel size.

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