ALGORITHMS FOR ARNOL’D TONGUES AND QUASI-PERIODIC TORI: A CASE STUDY

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Abstract
This paper presents algorithms for the computation of Arnol’d tongues and quasi-periodic curves in a two-parameter plane and demonstrates their performance with an example from nonlinear electrical engineering. Both methods are complementary and, together, allow the computation of the two-parameter bifurcation diagram near a one-parameter family of tori with parallel flow. The algorithms are simple to implement and compute Arnol’d tongues or quasi-periodic curves for user-defined rotation numbers.

Key words
quasi-periodic tori, phase lock, Arnol’d tongue, finite difference, continuation

1 Introduction
Quasi-periodic and phase-locked oscillations are typical phenomena in forced or coupled oscillators. Let us, as a motivating example, consider the parametrically forced system

$$\ddot{x} + \alpha \dot{x}^3 - \beta \dot{x} + (1 + B \sin 2t)x = 0. \quad (1)$$

Here, the parameters $\alpha = \varepsilon - B$ and $\beta = \varepsilon/2 - B$ are chosen such that the system’s response is an almost harmonic $2\pi$-periodic oscillation, in other words, the system is in 1:2 sub-harmonic resonance. A detailed one-parameter study of this example for $B = 0.1$ was carried out in [Schilder, Osinga and Vogt, 2004; Schilder, Vogt, Schreiber and Ozinga, 2005]. Here we present a two-parameter study, which gives a far more complete picture of the qualitative properties of the system’s response depending on the two parameters $B$ and $\varepsilon$.

For $B = 0$ equation (1) becomes autonomous and the resulting system has a limit cycle for $\varepsilon > 0$. Hence, for $B = 0$ and $\varepsilon > 0$ an $\varepsilon$-family of normally attracting invariant tori with parallel flow exists in the phase space $\mathbb{R}^2 \times (\mathbb{R}/\pi)$. The flow on these tori is called quasi-periodic if the ratio of the two internal frequencies (the rotation number) is irrational and resonant otherwise. Due to normal attractivity this $\varepsilon$-family of invariant tori persists for sufficiently small values of the forcing parameter $B$. However, the flow on the tori may change from parallel to phase-locked. More precisely, in the $(\varepsilon, B)$-parameter plane exist so-called Arnol’d tongues and the flow on corresponding tori is typically phase-locked [Aronson, Chory, Hall and McGehee, 1983; Glazier and Libchaber, 1988]. The complementary set of parameter values consists of smooth curves (quasi-periodic curves) such that the flow on the corresponding tori is quasi-periodic and the irrational rotation number satisfies certain number-theoretical properties. The union of these curves forms a Cantor-like set of positive measure and is nowhere dense. The quasi-periodic curves end at points where the Arnol’d tongues overlap and a direct transition from quasi-periodicity to chaotic behaviour may occur [Östlund, Rand, Sethna and Siggia, 1983].
Figure 1 shows the bifurcation diagram of system (1) in the $(\varepsilon, B)$ parameter plane. The blue shaded areas are the Arnol’d tongues for all rotation numbers in $L_4$, where $L_4$ was computed from $L_0 = \{1/2, 1/3, 1/4\}$ as follows. Starting with the ordered sequence $L_n := \{p_1/q_1, \ldots, p_m/q_m\}$ of rational numbers we obtain the next sequence $L_{n+1}$ by inserting all Farey sums $p_1/q_1 \oplus p_2/q_2 = (p_1 + p_2)/(q_1 + q_2)$ of two consecutive rational numbers into $L_n$. In the boundary of the 1:4 tongue is a small loop visible near $(\varepsilon, B) = (9, 0.6)$, which is explained in the next section.

The union of the areas A, B and C can be regarded as a degenerate 1:2 Arnol’d tongue. For parameters changing from region A over B to C, two consecutive period-doubling bifurcations of the zero solution occur. A 1:2 phase-locked torus exists in region C as a connecting manifold between the two 1:2 sub-harmonic solutions which branched off the zero solution. These two sub-harmonic solutions collide and disappear in a saddle-node bifurcation when leaving the region C. The red curves are branches of quasi-periodic tori with the irrational rotation numbers $\varphi = n\sqrt{2}/140 \text{ for } n = 47, 41, 36 \text{ and } 29$.

2 Computation of Arnol’d tongues

The typical way to compute a $p/q$ Arnol’d tongue is to follow the loci of period-$q$ saddle-node bifurcations which define its boundaries. This is efficient for smaller values of $q$, in particular for the strong resonances with $q = 1, 2, 3, 4$. However, this approach may fail for larger values of $q$ close to zero forcing, decoupling or a Neimark-Sacker curve, because the problem of locating the saddle-node point is then very ill posed [Schilder and Peckham, 2005].

The basic idea of our algorithm originates from the observation that an Arnol’d tongue is just a specific projection of a so-called resonance surface onto the parameter plane [McGehee and Peckham, 1994]. While the computation of the saddle-node curves may be ill posed, the computation of the resonance surface itself is always a well-posed problem. Since, close to decoupling, the resonance surface is topologically equivalent to a cylinder, we can construct a boundary value problem for a section of this cylinder and perform a one-parameter continuation to compute the resonance surface using, for example, AUTO [Doedel et al., 1997]. This algorithm is explained in detail in [Schilder and Peckham, 2005], including a discussion of adaptation and convergence.

Consider our example system (1) for which the natural choice is to compute the sections $B = \text{constant}$; see Fig. 1. We introduce the parametrisation $\gamma(s) := (\xi(s), \varepsilon(s))$, where $\xi = (x(0), \dot{x}(0))$ is a fixed point of the period-$q\pi$ stroboscopic map at $t = 0$ and $\varepsilon(s)$ is the parameter associated with the periodic solution through $\xi(s)$. Note that at the tip of the Arnol’d tongue we have $\lambda(s) = 0 = \text{constant}$, which causes the ill-posedness of the saddle-node computation close to zero forcing. Under the assumption that the curve $\gamma$ intersects the line $\xi_1 = 0$ transversally, a set of $N$ points on the curve $\gamma$ can be computed as a solution of the two-point boundary value problem

$$
\begin{align*}
\dot{\xi}_i &= q f(\xi_i, t, \varepsilon, B), \\
y_i(0) &= y_i(q\pi), \\
\|\xi_i - \xi_{i+1}\|_2 &= h, \\
(\xi_1) &= 0,
\end{align*}
$$

where $y = (x, \dot{x})$, $i = 1, \ldots, N$, and $f$ is the right-hand side of (1) rewritten as the first order system

$$
\dot{y} = f(y, t, \varepsilon, B).
$$

The first two equations in (2) constitute a boundary value problem for a $q\pi$-periodic solution of (3). The third equation requires that the points along $\gamma$ are equidistributed with respect to arc-length in the $(\xi_1, \xi_2)$ plane and the last condition fixes the initial point for $s = 0$ on $\gamma$. These are $3N + 1$ conditions for the unknowns $\xi_i, \varepsilon_i$ and $h$.

The solution of system (2) still depends on the free parameter $B$ and we can continue it using standard continuation codes for boundary value problems as, for example, AUTO [Doedel et al., 1997]. If we use a piecewise linear approximation for $\gamma$, then the boundaries of the computed Arnol’d tongues depending on
The characteristic field of the invariance condition (4) and the difference star of the second order leap-frog method. The light blue triangle indicates the numerical domain of dependence.

Note that this always underestimates the width of the tongue. If the approximated tongues overlap, then the exact tongues are guaranteed to overlap.

Our experience shows that the presented algorithm provides accurate approximations to high-period Arnold’s tongues already for a small number of mesh points. For this reason and to compare our results with computations of period-\(q\) saddle-node curves, we illustrate the convergence behaviour in Figs. 2 and 3 for the 1:3 and 1:4 tongues, the two widest tongues in Fig. 1. The black curves are the saddle-node curves and the coloured curves are the approximated boundaries for \(N = 10, 20, 30\) and 40 mesh points. Close to zero forcing, where the tongues are narrow, we observe uniform convergence. For larger values of \(B\), where the tongues become very wide, the convergence is less uniform and we even observe loops in the boundary of the 1:4 tongue. This is due to the increasingly complicated geometry of the curve \(\gamma\) in the \((\xi_1, \xi_2)\) plane. The used number of mesh points is too small to capture the geometry correctly and the distribution of these points along \(\gamma\) is therefore not unique.

### 3 Computation of quasi-periodic tori

A quasi-periodic invariant torus of our \(\pi\)-periodically forced system (3) is a solution of the partial differential equation

\[
2u_\theta + \omega u_\phi = f(u, \theta/2, \varepsilon, B),
\]

(\(\theta, \phi\) \(\in (\mathbb{R}/2\pi)^2\), the so-called invariance condition [Schilder, Osinga and Vogt, 2004]). A similar condition holds for quasi-periodic tori of maps [Jorba, 2001]. Here, \(\rho = 2/\omega\) is the irrational rotation number and the tuple \((2, \omega)\) is the frequency basis of the torus. Equation (4) must be supplemented by a phase condition, which fixes the first parameter \(\varepsilon\). The forcing amplitude \(B\) is used as the primary continuation parameter in our subsequent computations.

The invariance condition (4) is a hyperbolic partial differential equation system with same principal part on a periodic domain. This equation can be discretised in several ways, for example, by finite differences or using Fourier-Galerkin methods, two different approaches proposed in [Schilder, Osinga and Vogt, 2004; Schilder, Vogt, Schreiber and Osinga, 2005]. For the computations in this paper we used leap-frog finite-difference methods. The difference star of the method of order 2 is sketched in Fig. 4 together with the characteristic field of (4). The tip of the numerical domain of dependence is indicated by the blue shaded triangle. As one continues this domain backwards, it eventually covers the entire torus. Hence, the CFL condition does not impose a condition on the step-sizes used in the discretisation. However, there is strong numerical evidence that the mesh should be aligned to the characteristics as indicated. The condition number of the linearised system was always found to be minimal for such meshes.

As an example we continue the quasi-periodic tori of system (1) starting at \(B = 0\) for the rotation numbers \(\rho = n\sqrt{2}/140 \approx n/100\) with \(n = 47, 41, 36\) and 29, indicated by the red curves from left to right in Fig. 1, respectively. During the continuation we monitor an estimation of the error of our approximate solutions, which is computed as the difference of the numerical solutions of an order 4 and an order 2 method. The convergence behaviour depending on \(B\) along the quasi-periodic curve for \(n = 41\), which is the second from left in Fig. 1, is illustrated in Fig. 5. The figure shows that the estimated error decreases uniformly as the number of mesh points is increased. On the other hand, the error becomes larger for increasing values of \(B\), which may indicate a loss of smoothness.

Some of the tori along the branch \(n = 41\) are depicted in Fig. 6 in the unwinded phase space \(\mathbb{R}^2 \times [0, \pi]\). The first torus in panel (a) is obtained for zero forcing and is just a cylinder. The tori develop a nontrivial shape as the forcing amplitude \(B\) increases. The torus in Fig. 6 seems to be close to break-up and the appearing ripples suggest that the torus is no longer smooth.
Figure 6. The invariant torus along the second from left branch in Fig. 1 in the unwinded phase space $\mathbb{R}^2 \times [0, \pi]$ for $B \approx 0.000, 0.312, 0.395$ and $0.499$, panel (a)–(d), respectively. In the unforced case, the torus is the cylinder depicted in panel (a). As the forcing increases, the shape of the tori becomes more complicated as shown in panels (b) and (c). The last torus in panel (d) seems to be close to break-up and the appearing ripples indicate a loss of smoothness.

4 Conclusion

In this paper we discussed two algorithms for the numerical analysis of resonance phenomena, the occurrence of sub-harmonic resonances and of quasi-periodicity. Both algorithms are complementary and allow the computation of the full bifurcation diagram close to zero forcing or decoupling, respectively. The performance of both algorithms was demonstrated with the computation of the two-parameter bifurcation diagram of a parametrically forced system. The computed bifurcation diagram provides a complete overview of the qualitative behaviour close to zero forcing. For larger forcing amplitudes additional bifurcations are to be expected within the Arnol’d tongues.

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References


