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Visualizing the transition to chaos in the Lorenz system

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Summary. The Lorenz system still fascinates many people because of the simplicity of the equations that generate such complicated dynamics on the famous butterfly attractor. This paper addresses the role of the global stable and unstable manifolds in organising the dynamics. More precisely, for the standard system parameters, the origin has a two-dimensional stable manifold and the other two equilibria each have a two-dimensional unstable manifold. The intersections of these manifolds in the three-dimensional phase space form heteroclinic connections from the nontrivial equilibria to the origin. A parameter-dependent visualization of these manifolds clarifies the transition to chaos in the Lorenz system.

1 Introduction

The Lorenz system is arguably the most famous dynamical system exhibiting chaotic dynamics. When Lorenz examined the behaviour of Rayleigh-Bénard convection in 1963, he found that for Rayleigh numbers well after the critical value of onset of convection the behaviour of the fluid is aperiodic. Since the motion is nevertheless bounded, he could prove that an attractor must exist along with infinitely many unstable periodic orbits; see [Lo63]. In his paper Lorenz uses extremely simplified equations by considering only one mode for the velocity and two modes for the temperature of the fluid. This results in the system now know as the Lorenz system

\[
\begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= \rho x - y - xz, \\
\dot{z} &= xy - \beta z.
\end{align*}
\]

Here, the main parameter is the Rayleigh number \( \rho \), and system (1) is rescaled such that the onset of convection occurs at \( \rho = 1 \). Lorenz provides physically
relevant values for the parameters in [Lo63], namely, the now classic choice
\[\alpha = 10, \quad \gamma = 28, \quad \beta = \frac{8}{3}.\]

Chaos in the Lorenz system is typically visualized by integrating from an
arbitrary initial condition for a sufficiently long (positive) time, discarding
the transient and displaying the resulting orbit in \((x, y, z)\)-space — the phase
space of the Lorenz system. The image is a representation of the ‘butterfly
attractor’ or Lorenz attractor, which is a fractal object on which the dynamics
is chaotic. That this is indeed the case for the classic parameter values above
was, in fact, proven only recently [Tu99, Vi00].

In this paper we present a different way of visualizing the chaotic dynamics.
Namely, we consider the stable manifold \(W^s(0)\) of the origin, which is
a smooth two-dimensional surface in phase space with a very intricate geo-
metry. It contains all initial conditions that converge to the origin under for-
ward integration. The origin is a saddle equilibrium of (1) with two stable
(contracting) and one unstable (repelling) eigenvalue; see also [GH86]. The
Stable Manifold Theorem [PdM82] then guarantees the existence of \(W^s(0)\)
as an immersed manifold that is tangent to, and has the same dimension as,
the stable eigenspace associated with the two stable eigenvalues. The stable
manifold \(W^s(0)\), which we refer to as the Lorenz manifold, provides a global
impression of how chaos is organised in phase space. In a way, the chaotic
dynamics is confined to the complement \(\mathbb{R}^3 \setminus W^s(0)\).

The visualization of the Lorenz manifold is quite a challenge. Topologic-
ally, any finite piece of \(W^s(0)\) is a disk, but one whose embedding in \(\mathbb{R}^3\)
becomes increasingly complicated geometrically. The first images of \(W^s(0)\)
are the hand-drawn sketches by Perelló from 1979 in [Pe79]. His work formed
the basis for the drawings of Shaw in [AS85]. In fact, the computation of
two-dimensional invariant manifolds in dynamical systems is an active field
of research, and several methods are available today to compute \(W^s(0)\)
umERICALLY; see [KOD+05]. Our own method [KO03, EKO07] constructs the
Lorenz manifold (or indeed other two-dimensional global invariant manifold)
as a set of geodesic level sets at appropriate distances from each other. It starts
from a disk in the stable eigenspace and chooses an appropriate geodesic dis-
tance of the next level set based on the local curvature of the manifold. The
mesh points on the new geodesic level set are constructed pointwise by solv-
ing a two-point boundary value problem; see also [KO07] in this volume, and
[KO03, EKO07] for full details.

Our geometrical way of building up a two-dimensional surface has advant-
ages in terms of visualizing and then understanding the nature of the Lorenz
manifold; see also [OK02, KO04] for more images and animations. Here, we
take the visualization of the Lorenz manifold further by considering how the
organisation of chaos varies with the Rayleigh number \(\gamma\); see [DKO06] for
more mathematical details. Specifically, we show in Sect. 2 the transition of
the Lorenz manifold through the first homoclinic bifurcation at \(\gamma \approx 13.9162,\)
which is also known as a homoclinic explosion. In this bifurcation all complic-
ated dynamics is generated. We then show in Sect. 3 how the Lorenz manifold
interacts with the unstable manifolds of saddle-periodic orbits and secondary equilibria of the Lorenz system when \( \rho \) is increased up to the classic value of \( \rho = 28 \). We summarize our findings in Sect. 4.

2 Transition through the homoclinic explosion

At the homoclinic explosion at \( \rho \approx 13.9162 \) each branch of the one-dimensional unstable manifold of the origin forms a homoclinic loop after a single rotation around one of the two secondary equilibria

\[
p^\pm = (\pm \sqrt{\beta (\rho - 1)}, \pm \sqrt{\beta (\rho - 1)}, \rho - 1).
\]

As is well known, infinitely many periodic orbits are created in this bifurcation, including a pair of saddle periodic orbits \( \Gamma^\pm \); see [Sp82, DKO06]. Note that the Lorenz equations have the symmetry of rotation by \( \pi \) around the \( z \)-axis; indeed \( p^+ \) and \( p^- \) and \( \Gamma^+ \) and \( \Gamma^- \) are each other’s symmetric counterparts, and the Lorenz manifold has this rotational symmetry.

We are interested here in visualizing how this homoclinic bifurcation influences the geometry of \( W^s(0) \). Figure 1 shows \( W^s(0) \) just before, and just after the homoclinic explosion, namely at \( \rho = 13 \) and at \( \rho = 15 \), respectively. The manifold \( W^s(0) \) has been computed up to geodesic distance 100 and it is rendered transparent. Furthermore, to highlight its geometric structure the level sets from 97 to 100 are given a different (transparent) colour. The vertical axis in the pictures is the \( z \)-axis and the origin is in the middle. All images in this paper were produced with the package Geomview [PLM93].
Figure 1(a) illustrates the situation for $\varrho < 13.9162$, namely for $\varrho = 13$. In this case the Lorenz manifold wraps around $p^\pm$ once, returning back near 0 at a negative $z$-value, after which it folds down (towards negative $z$) such that it lies practically flat against the lower half-disk of $W^s(0)$. At the same time, there is a helix along the positive $z$-axis. The purpose of the visualization in Fig. 1 is that one can follow the coloured outer band of $W^s(0)$ to obtain an idea of the overall geometry of the surface. To this end, consider starting at the lowest point (0, 0, −100) and moving to the left. The outer band turns up and around before making a relatively sharp turn slightly up and back down towards the negative $z$-axis. With some practice, one can observe that the band continues further to the right and starts winding its way up the helix until it reaches (0, 0, 100). When moving right from (0, 0, −100) the symmetrical behaviour can be observed.

The situation for $\varrho > 13.9162$ is visualized in Fig. 1(b) for $\varrho = 15$. As one follows the outer band to the left from the lowest point (0, 0, −100) the initial behavior is as before; indeed this first part of the outer band is virtually identical to that shown in Fig. 1(a). However, where the band in Fig. 1(a) suddenly turns slightly up and back down towards the negative $z$-axis, in Fig. 1(b) the relatively sharp turn is in the exact opposite direction and the band never passes near the negative $z$-axis before reaching (0, 0, 100).

It is important to realize that this dramatic change in the geometry of the geodesic level sets is not due to the relatively large gap between the two $\varrho$-values, 13 and 15, before and after the homoclinic bifurcation. The switch between passing by the negative $z$-axis again or not is sudden and immediate and a result of the existence of the homoclinic loop at $\varrho \approx 13.9162$. In fact, exactly at the homoclinic explosion, the Lorenz manifold returns to and ‘closes up’ in a non-smooth way along a special curve known as the strong stable manifold of the origin; see [DKO06] for details.

### 3 Intersections of two-dimensional manifolds

In the homoclinic explosion two stable periodic orbits $\Gamma^\pm$ are created, which are of saddle type. Therefore, they come with two-dimensional unstable manifolds $W^u(\Gamma^\pm)$. Due to the symmetry of the Lorenz system it suffices to compute only, say, $W^u(\Gamma^+)$. As $\varrho$ is increased the periodic orbits $\Gamma^\pm$ shrink down in size. The Lorenz attractor is created in a heteroclinic bifurcation between the origin and $\Gamma^\pm$, which takes place at $\varrho \approx 24.0579$. Finally, the periodic orbits disappear in a subcritical Hopf bifurcation of the secondary equilibria $p^\pm$ at $\varrho \approx 24.7368$. As a consequence, $p^\pm$ lose their stability and become saddles with two-dimensional unstable manifolds $W^u(p^\pm)$. The manifolds $W^u(p^+)$ and $W^u(\Gamma^+)$ consists of infinitely many layers that are extremely close together. By identifying these layers one obtains a branched surface known as the template of the Lorenz attractor, which can be used to describe the chaotic dynamics in the Lorenz system [GHS97, GL04, Wi98].
Fig. 2. Transparent renderings of the Lorenz manifold $W^s(0)$ computed up to geodesic distance 100 as it intersects the unstable manifolds $W^u(\Gamma^\pm)$ and $W^u(p^\pm)$, respectively, for $\varrho = 15$ (a), for $\varrho = 19$ (b), for $\varrho = 23$ (c), and for $\varrho = 28$ (d).

The key in the transition from the homoclinic explosion at $\varrho \approx 13.9162$ to the classic situation for $\varrho = 28$ is to understand how the Lorenz manifold intersects $W^u(p^\pm)$ and $W^u(\Gamma^\pm)$, respectively. To compute a suitable first part of $W^u(p^+)$ or $W^u(\Gamma^+)$, which effectively represents the template, we continue an orbit segment that starts on a fixed vector in the unstable eigenspace of $p^+$ and ends in the section $\{z = \varrho - 1\}$ either near $p^+$ or near $p^-$. This continuation is done with the package AUTO [Do81, CCF+97]; see also [DKO06] for a more detailed description of this method.

Figure 2 shows the Lorenz manifold $W^s(0)$ for four values of $\varrho$ as it intersects the unstable manifolds $W^u(\Gamma^\pm)$ (panels (a)–(c)) and $W^u(p^\pm)$ (panel (d)). In these images $W^s(0)$ is rendered transparent, while $W^u(\Gamma^\pm)$ and $W^u(p^\pm)$ are shown as solid surfaces. In this way, one gets an impression of the
intersection of the two surfaces. The viewpoint in Fig. 2 is fixed and chosen exactly as for Fig. 1; compare Fig. 2(a) with Fig. 1(b). In particular, the vertical axis is again the $z$-axis. Figure 2(a)–(c) shows $W^u(Γ^±)$ for $ϱ = 15$, $ϱ = 19$ and $ϱ = 23$, respectively. Notice how $W^u(Γ^±)$ grows while the periodic orbits $Γ^±$ actually shrink. They surround the secondary equilibria $p^±$, which are attractors. Finally, in Fig. 2(d) the periodic orbits are gone and the image shows the manifolds $W^u(p^±)$ of the equilibria $p^±$, which are now saddle points. We remark that this transition from $W^u(Γ^±)$ to $W^u(p^±)$ is smooth (i.e., is not noticeable on the level of these two-dimensional surfaces). Notice further that, as $ϱ$ increases, the amount of spiralling of the helix around the positive $z$-axis decreases.

4 Conclusions

We studied the transition to chaos in the Lorenz system by considering two-dimensional global stable and unstable manifolds. We first showed how the Lorenz manifold, the stable manifold of the origin, changes during the homoclinic bifurcation that gives rise to the complicated dynamics of the Lorenz system. We then considered how it intersects the unstable manifolds of the bifurcating periodic orbits and secondary equilibria.

Crucial for such a study is the ability to visualize these complicated geometric objects efficiently. In the case of the Lorenz manifold this is helped by the fact that we build it up from geodesic level sets. In particular, this allows us to highlight certain level sets to emphasize particular aspects of the geometry of the object.

References


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