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Stiffness of Clays and Silts: Modeling Considerations

P. J. Vardanega, Ph.D. M.ASCE\textsuperscript{1} and M. D. Bolton, Ph.D.\textsuperscript{2}

ABSTRACT

A large database has recently been published that details the development of new empirical expressions for the stiffness reduction with strain of clays and silts. In this note, the same database is used to examine two major considerations for engineers using these expressions in numerical analyses: the transformation from secant to tangent stiffness and the effect of stress history.

Keywords: Stiffness; Clays; Silts; Design; Deformations; Modelling; Statistical Analysis

\textsuperscript{1} Lecturer in Civil Engineering, Department of Civil Engineering, University of Bristol, Bristol, BS8 1TR, U.K., formerly Research Associate, Department of Engineering, University of Cambridge, Cambridge, CB2 1PZ, U.K. (Corresponding Author) Email: p.j.vardanega@bristol.ac.uk

\textsuperscript{2} Professor of Soil Mechanics, Department of Engineering, University of Cambridge, Cambridge, CB2 1PZ, U.K. Email: mdb8@cam.ac.uk
INTRODUCTION

The estimation and measurement of soil modulus reduction with increasing strain has been the subject of much research in geotechnical engineering (e.g. Kondner, 1963; Hardin and Drnevich, 1972a, 1972b; Vucetic and Dobry, 1991; Fahey, 1992; Fahey and Carter, 1993; Stokoe et al. 1994; Stokoe et al. 1999; Hardin and Kalinski, 2005 and Gasparre et al. 2007; Oztoprak and Bolton, 2013 and Wichmann and Triantafyllidis, 2013a, 2013b). The importance of understanding small-strains for geotechnical design has been discussed extensively in Burland (1989) and Atkinson (2000).

Vardanega and Bolton (2013) have recently published a large database that was used to derive simple empirical expressions for modulus reduction for clays and silts. The substantive details of the database formulation, the sources of data, and their subsequent analysis will not be repeated here. Figure 1 shows the Casagrande plot for the soils in the database: a variety of fine-grained soil types are represented.

**Figure 1: Casagrande plot of the soils in the database presented in Vardanega and Bolton (2013) (chart design adapted from Casagrande, 1947; Howard, 1984; and BS5930 British Standards Institution, 1999)**
Static and Dynamic adjustments

The stiffness of fine grained soils is well-known to be rate sensitive (e.g. Richardson and Whitman, 1963). Vardanega and Bolton (2013) presented calibrated empirical expressions [based on the general form adopted in Darendeli (2001)] demonstrating that rate-effect adjustments are necessary when comparing data tested in different apparatuses. The new curves were compared with those of Vucetic and Dobry (1991) which do not explicitly account for rate effects, and which are now seen to be too widely spaced.

The database presented in Vardanega and Bolton (2013) had the original test data from 10 publications (67 tests) adjusted for rate effects to two representative strain rates, namely $10^{-6}/s$ and $10^{-2}/s$, with the former attempting to simulate a standard triaxial test and the latter simulating a standard earthquake. This adjustment was based on the assumption of a stiffness variation of 5% per factor 10 on strain rate, providing an indication of the increase in stiffness that is implied when moving from $10^{-6}/s$ (static adjustment) to $10^{-2}/s$ (dynamic adjustment) in these two design situations.

Calibrated Stiffness Reduction Functions

The newly calibrated functions to describe the modulus reduction of clays and silts from Vardanega & Bolton (2013), and the prediction of the reference strain parameter ($\gamma_{\text{ref}}$) are as follows, for the database with the static adjustment applied:

\[
\frac{G}{G_{\text{max}}} = \frac{1}{1 + \left( \frac{\gamma}{\gamma_{\text{ref}}} \right)^{0.74}}
\]

(1a)

where,

\[
\gamma_{\text{ref}} = 2.2 \left( I_P/1000 \right) \quad (I_P \text{ expressed numerically and not as a percentage})
\]

(1b)

For the database with the dynamic adjustment applied:
\[
\frac{G}{G_{\text{max}}} = \frac{1}{1 + \left(\frac{\gamma}{\gamma_{\text{ref}}}\right)^{0.94}}
\]  
(2a)

where,

\[
\gamma_{\text{ref}} = 3.7 \left(\frac{I_p}{1000}\right) \quad (I_p \text{ expressed numerically and not as a percentage})
\]  
(2b)

In this note, the same database is used to examine two major considerations for engineers using these expressions in numerical analyses: (a) the transformation from secant to tangent stiffness and (b) the effect of stress history.

**SMALL STRAIN REGION**

The reduction of the shear stiffness of a soil with increasing strain from its purely elastic maximum value \( G_{\text{max}} \) is sketched in Figure 2 for both monotonic and cyclic tests. Referring to Figures 2 and 3 we can say that \( G_{\text{max}} = G_{\text{sec}} = G_{\text{tan}} \) in the linear elastic strain range and that at greater strains one may describe the modulus either as a secant \( (G_{\text{sec}}) \) or a tangent \( (G_{\text{tan}}) \). The use of \( G_{\text{sec}} \) rather than \( G_{\text{tan}} \) is preferred in the processing of test data since it is an order of magnitude less influenced by random errors (noise). Nevertheless, \( G_{\text{tan}} \) is preferred in numerical procedures which require the assembly of an incremental stiffness matrix.

![Figure 2: Definitions of secant stiffness \( G, G_{\text{max}}, G_{\text{cyclic}} \)](image)
TANGENT STIFFNESS

If the tangent stiffness is desired, for numerical analysis, then it can easily be calculated from the secant stiffnesses that are quoted Vardanega and Bolton (2013), which will consistently be referred to below simply as $G$. Given that equations (1a) and (2a) have the same form [the form used in Darendeli (2001)], one can write:

$$ G = \frac{\tau}{\gamma} \quad G_{\tan} = \frac{d\tau}{d\gamma} $$

By definition:

$$ \tau = G \gamma $$

Differentiating equation (4) with respect to strain:

$$ G_{\tan} = \frac{d\tau}{d\gamma} = G + \gamma \frac{dG}{d\gamma} $$

By differentiating equation (3):

$$ \tau = \frac{1}{G_{\max} \left(1 + \left(\frac{\gamma}{\gamma_{ref}}\right)\right)} $$
Substituting equation (6) in equation (5) and reorganising, one obtains:

$$\frac{G_{\text{tan}}}{G} = 1 - \frac{\alpha}{\left[\left(\frac{\gamma_{\text{ref}}}{\gamma}\right)^{\alpha} + 1\right]}$$

From equation (7) it can be seen that when $\alpha = 0.74$ (static adjustment):

$$\gamma = 0 \quad G_{\text{tan}} = G_{\text{max}} = G$$

$$\gamma = \gamma_{\text{ref}} \quad G_{\text{tan}} = G\left[1 - \left(\frac{\alpha}{2}\right)\right] = 0.63G$$

$$\gamma = 10\gamma_{\text{ref}} \quad G_{\text{tan}} = G\left[1 - \left(\frac{\alpha}{1 + 0.1^{\alpha}}\right)\right] = 0.37G$$

From equation (7) it can be seen that when $\alpha = 0.94$ (dynamic adjustment):

$$\gamma = 0 \quad G_{\text{tan}} = G_{\text{max}} = G$$

$$\gamma = \gamma_{\text{ref}} \quad G_{\text{tan}} = G\left[1 - \left(\frac{\alpha}{2}\right)\right] = 0.53G$$

$$\gamma = 10\gamma_{\text{ref}} \quad G_{\text{tan}} = G\left[1 - \left(\frac{\alpha}{1 + 0.1^{\alpha}}\right)\right] = 0.16G$$

Larger values of $\alpha$ produce a faster diminution in $G$ with strain through equation (3), and even more so in $G_{\text{tan}}$ through equation (7).

CONSIDERATION OF STRESS HISTORY

Database Variability

Table 1 shows the 67 tests that comprised the database presented in Vardanega and Bolton (2013) on 21 clays and silts re-classified according to their stress history. Twenty-four of the tests were on soils that were able to be classified as normally or lightly overconsolidated (OCR $< \approx 2$). Twenty-six of the tests were on soils that were able to be classified as heavily overconsolidated (OCR $> \approx 2$).
Table 1: Stress History Categorization of the Database Presented in Vardanega & Bolton (2013) along with Average Values of the Curvature Parameter ($\alpha$) and Reference Strain ($\gamma_{ref}$)

<table>
<thead>
<tr>
<th>Publication</th>
<th>Soils Tested</th>
<th>No. of Tests</th>
<th>Average $\alpha_{stat}$</th>
<th>Average $\alpha_{dyn}$</th>
<th>Average $\gamma_{ref,stat}$</th>
<th>Average $\gamma_{ref,dyn}$</th>
<th>Notes on overconsolidation ratio</th>
<th>Classification of the soil deposit based on overconsolidation ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anderson and Richart (1976)</td>
<td>Leda Clay, Detroit Clay, Ford Clay, Santa Barbara Clay and Eaton Clay</td>
<td>5</td>
<td>0.65</td>
<td>0.96</td>
<td>0.00065</td>
<td>0.0012</td>
<td>Insufficient information available</td>
<td>Unclassified</td>
</tr>
<tr>
<td>Kim and Novak (1981)</td>
<td>Seven Ontario fine grained soils (low plasticity)</td>
<td>12</td>
<td>0.82</td>
<td>1.25</td>
<td>0.00036</td>
<td>0.00057</td>
<td>Natural OCR ranges from 1.8 to 6.8. Testing done at confining stresses $&gt;&gt; p’$ in-situ</td>
<td>Unclassified</td>
</tr>
<tr>
<td>Georgiannou et al. (1991)</td>
<td>Pietrafitta, Vallericca and Todi Clay</td>
<td>6</td>
<td>0.74</td>
<td>1.33</td>
<td>0.00065</td>
<td>0.00099</td>
<td>Authors state that the clays are overconsolidated. Probably heavily over-consolidated given that the natural condition of the clays are likely to be similar to those studied by Rampello and Silvestri (1993).</td>
<td>Heavily overconsolidated</td>
</tr>
<tr>
<td>Rampello and Silvestri (1993)</td>
<td>Pietrafitta and Vallericca Clay</td>
<td>4</td>
<td>0.69</td>
<td>1.27</td>
<td>0.00062</td>
<td>0.00085</td>
<td>OCR values ~ 4.0 &amp; 4.4</td>
<td>Heavily overconsolidated</td>
</tr>
<tr>
<td>Shibuya and Mitachi (1994)</td>
<td>Hachirōgata Clay</td>
<td>7</td>
<td>0.65</td>
<td>1.07</td>
<td>0.0021</td>
<td>0.0036</td>
<td>The authors stated that the clay deposit was not believed to have been subjected to mechanical overconsolidation</td>
<td>Normally consolidated</td>
</tr>
<tr>
<td>Soga (1994)</td>
<td>San Francisco Bay Mud (3m and 5m deep samples)</td>
<td>3</td>
<td>0.57</td>
<td>0.76</td>
<td>0.0012</td>
<td>0.0024</td>
<td>OCR ~ 1.5 at 5m depth</td>
<td>Lightly overconsolidated</td>
</tr>
<tr>
<td>Author and Reference</td>
<td>Clay Type</td>
<td>OCR</td>
<td>SR</td>
<td>GR</td>
<td>OCR Value</td>
<td>Overconsolidation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------------</td>
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<td>----</td>
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<td>-----------</td>
<td>-------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pisa Clay (Horizon A)</td>
<td>1 0.71 0.93 0.00039 0.00078</td>
<td>OCR ~ 4.5 (4m sample)</td>
<td>Heavily overconsolidated</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pisa Clay (Horizon B)</td>
<td>4 0.74 0.91 0.00068 0.0012</td>
<td>OCR average ~ 1.3 (varies from 1.2 to 1.5) (Sampling from 10 to 19m)</td>
<td>Lightly overconsolidated</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Doroudian and Vucetic (1999)</td>
<td>Highly plastic Santa Barbara Silt</td>
<td>4 0.82 1.13 0.00088 0.0017</td>
<td>OCR = 17 at 31.0m depth (sampling from 9.5 to 64.6m)</td>
<td>Heavily overconsolidated</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yimsiri (2001)</td>
<td>London Clay</td>
<td>6 0.74 0.87 0.0013 0.0024</td>
<td>Ancient Eocene clay overlain by approximately 360m of submerged sediments (Bishop et al. 1965) in the Wraysbury district (Hight et al. 2007). Chandler (2000) using geological evidence concluded that overburden removed ~ 200 m</td>
<td>Heavily overconsolidated</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teachavoraskinskun et al. (2002)</td>
<td>Bangkok Clay</td>
<td>10 0.77 0.88 0.00090 0.0016</td>
<td>No specific OCR details given in the paper. However, Tanaka et al. (2001) give a value of 1.3 and Sambhandharaska et al. (2003) give values of OCR ~ 1.5-2.1 for the depth range 8.5 to 4m (similar depth to the data studied)</td>
<td>Lightly overconsolidated</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gasparre (2005)</td>
<td>London Clay</td>
<td>5 0.91 0.98 0.0019 0.0028</td>
<td>See description of London Clay above</td>
<td>Heavily overconsolidated</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Seventeen of the tests could not be classified in either category. [In the case of the data from Anderson and Richart (1976) insufficient information was provided about the natural soil deposits. In the case of the data from Kim and Novak (1981) there was apparently no attempt to replicate in-situ conditions for the tests studied.]

Table 2 shows that the difference between the average curvature parameters for the three classifications is very small. This trend holds both for the database with the static adjustment and with the dynamic adjustment applied. Table 2 also demonstrates that the average value of the reference strain is not greatly different between the normally and lightly overconsolidated category and the heavily overconsolidated category. Vardanega and Bolton (2013), following the work of Vucetic and Dobry (1991), showed that $\gamma_{ref}$ is a strong function of plasticity index. The static and dynamic adjustments also show that rate effects will have a significant effect on the reference strain. However, it would now appear that there is no significant influence of OCR on the reference strain. Figure 4 shows equation (3) plotted with the average value of $\alpha_{stat}$ for the whole database denoted as ‘$\alpha_{stat}(\text{average})$’ also plotted is equation (3) with values of $\alpha_{stat} \pm 1$ standard deviation, denoted as ‘$\alpha_{stat}(\text{plus 1 SD})$’ and ‘$\alpha_{stat}(\text{minus 1 SD})$’ respectively. Also plotted is equation (3) with the average $\alpha_{stat}$ values shown in Table 2 for the normal and lightly overconsolidated classified soils and the heavily overconsolidated soils, denoted as ‘$\alpha_{stat}(\text{OCR < 2})$’ and ‘$\alpha_{stat}(\text{OCR > 2})$’ respectively. The upper and lower bounds of the normalised database presented in Vardanega and Bolton (2013) are also shown.

The influence of OCR on the curvature parameter ($\alpha$) does not appear to be significant, simply from a visual inspection of Figure 4. Similar trends are found using the database when the dynamic adjustment is applied.

It might be noted that the values of the average curvature parameters for the whole database are very similar to the average $\alpha$ values used in equation (1a) and equation (2a) but
they are not identical since the number of available data points varies between the individual test curves. The selection of the best-fit regression line to determine the $\alpha$ value ensures the maximum reduction of scatter.

Table 2: Summary of Average $\alpha$ Values and $\gamma_{ref}$ Values for the Three Stress History Categories

<table>
<thead>
<tr>
<th>Classification based on overconsolidation ratio</th>
<th>No. of tests in category</th>
<th>Average $\alpha_{stat}$</th>
<th>Average $\alpha_{dyn}$</th>
<th>Average $\gamma_{ref,stat}$</th>
<th>Average $\gamma_{ref,dyn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normally consolidated and lightly over-consolidated soils</td>
<td>24</td>
<td>0.70</td>
<td>0.93</td>
<td>0.0013</td>
<td>0.0022</td>
</tr>
<tr>
<td>Heavily over-consolidated soils</td>
<td>26</td>
<td>0.77</td>
<td>1.10</td>
<td>0.0011</td>
<td>0.0017</td>
</tr>
<tr>
<td>Unclassified soils</td>
<td>17</td>
<td>0.77</td>
<td>1.17</td>
<td>0.00045$^a$</td>
<td>0.00074$^a$</td>
</tr>
<tr>
<td>All tests</td>
<td>67</td>
<td>0.75$^b$</td>
<td>1.06</td>
<td>0.00097</td>
<td>0.0017</td>
</tr>
</tbody>
</table>

$^a$ Low average $\gamma_{ref}$ values due to the 12 tests on the low plasticity Ontario fine grained soils (Vardanega and Bolton, 2013). Also note that $\gamma_{ref}$ is strongly correlated with $I_p$ (Vardanega and Bolton, 2013).

$^b$ Standard deviation of $\alpha$ for the whole database ~0.12 (Vardanega and Bolton, 2013)

Figure 4: Variation of the curvature parameter ($\alpha$) within the database (static adjustment applied)
**Kinematic Yielding**

The apparently marginal difference between lightly and heavily overconsolidated clays, in regard to their normalised stress-strain curves, deserves further comment. Figure 5 is based on the kinematic yielding model of Jardine (1992) and Smith et al. (1992). Normally consolidated soil in situ can be represented by a point such as O in Figure 5, standing on some plastic yield surface labelled Y. Outward-directed stress paths would cause plastic hardening and would create positive excess pore pressures in undrained tests. Inward-directed stress paths, such as those involved in field sampling and core extrusion in the laboratory, would initially involve linear and then non-linear strains as the Y yield surface is dragged down towards the $p'$ axis. The location of the Y yield surface may, however, cause the unloading stress path to create some irrecoverable hardening before the $p'$ axis is reached. The state of isotropic stress at the outset of a standard triaxial compression test on a sample core may therefore be some point such as A in Figure 5, consistent with a new Y yield surface marked “disturbed” on the diagram. The fine grained soils reported in the database as being normally consolidated in situ will generally have been tested in shear after isotropic relaxation to a point such as A. If the sample is isotropically overconsolidated from A it will achieve some point B prior to the shear phase of the test, as will clays which are naturally overconsolidated in situ.

An undrained triaxial compression test from either A or B will initially involve the same process of kinematic yielding at constant $p'$ inside the Y yield surface. This is represented by the dragging upwards of the Y yield surface from points A or B, as shown in Figure 5. According to Jardine (1992) both stress paths should begin with similar stress-strain relations consistent with a kinematic hardening rule. Equation 3 can be regarded as an empirical expression of this proposition. If a constitutive modeller wished to propose that kinematic hardening be described by a unique expression, irrespective of stress history, then single
values would be required of exponent $\alpha$ in equation 3 and a constant coefficient (i.e. the $J$ value) linking reference strain ($\varepsilon_{ref}$) and plasticity index ($I_P$) in equations (1b) and (2b) for the strain-rate of interest.

At larger strains the influence of OCR has been shown to be significant (e.g. Vardanega et al. 2012). The findings of this note pertain to the small strain region.

![Figure 5: Kinematic yielding representation](image)

**SUMMARY REMARKS**

The following summary points are made based on the work described in this note:

(a) When performing numerical analysis the secant stiffness shear strain functions can be easily converted to tangent stiffness expressions: the curvature parameter ($\alpha$) is directly linked to the diminishing stiffness with increased strain, even more so in tangent stiffness expressions.

(b) Considering the fine-grained soils that could be classified as either normally or lightly overconsolidated and comparing them with the more heavily overconsolidated soils, it has
been demonstrated that the normalised stress-strain curves of these two categories of geological materials may be quite similar in tests starting from a condition of isotropic effective stress. This has been explained as being indicative of a kinematic hardening function that is relatively insensitive to the initial mean effective stress within the state boundary surface (the $Y_3$ yield surface), at least in the small-strain region which is the focus of this paper. It must be remembered, of course, that the in situ stress state will in general have an effective stress ratio $K_0 \neq 1$, and that geotechnical processes in the field will generally involve more diverse stress paths than, for instance, simple triaxial compression, copious data of which are uniquely available in the literature. The influence of $K_0$ and of stress-path, in other words the influence of anisotropy, on the shapes of stress-strain curves lies outside the scope of this note.

ACKNOWLEDGEMENTS

The authors thank the Cambridge Commonwealth Trust and Ove Arup and Partners for financial support to the first author during his doctoral studies. The authors also thank the editors and reviewers for their helpful suggestions and comments.

NOTATION

$G = \text{secant shear stiffness (see also } G_{\text{sec}})$

$G_{\text{cyclic}} = \text{secant shear stiffness measured in a cyclic test}$

$G_{\text{max}} = \text{shear stiffness at very small strains (sometimes referred to as } G_0)$

$G_{\text{sec}} = \text{secant shear stiffness (see also } G)$

$G_{\tan} = \text{tangent shear stiffness}$

$I_P = \text{plasticity index}$

$K_0 = \text{coefficient of earth pressure at rest}$

$p' = \text{mean effective stress}$

$q = \text{deviator stress}$
$SD$ = standard deviation

$w_L$ = liquid limit

$\alpha$ = curvature parameter in the modified hyperbolic equation

$\alpha_{dyn}$ = curvature parameter obtained when the fitting function is applied to data that had the dynamic adjustment applied (described in Vardanega and Bolton, 2013)

$\alpha_{stat}$ = curvature parameter obtained when the fitting function is applied to data that had the static adjustment applied (described in Vardanega and Bolton, 2013)

$\gamma$ = shear strain

$\gamma_{cyclic}$ = shear strain amplitude measured in a cyclic test

$\gamma_{ref}$ = reference strain equal to the shear strain at $0.5G_{max}$

$\gamma_{ref,dyn}$ = reference strain for a test (or series of tests) where the data had the dynamic adjustment applied as described in Vardanega and Bolton (2013) to account for rate effects

$\gamma_{ref,stat}$ = reference strain for a test (or series of tests) where the data had the static adjustment applied as described in Vardanega and Bolton (2013) to account for rate effects

$\tau$ = shear stress

**REFERENCES**


