THE ULTIMATE STRENGTH OF UNIFORMLY LOADED LATERALLY RESTRAINED RECTANGULAR TWO-WAY CONCRETE SLABS

by

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CONTAINS

PULLOUTS
The continuous encouragement of Professor Sir Alfred Pugsley during the course of this project is gratefully acknowledged. Gratitude is expressed to members of the academic staff of this University and to members of the computing staff of the University of Southampton for their valuable assistance. Thanks are due to Mr. H. Peart and to the other members of the technical staff for their help and cooperation during the experimental programme. Particular thanks are due to Miss M. Malone for careful typing of the manuscript, and to Mr. F. G. Garraway and Mrs. P. J. Smith for preparing the photographs and printing the figures.
SUMMARY

The thesis describes a theoretical and experimental investigation into the ultimate flexural strength of uniformly loaded two-way rectangular concrete slabs (reinforced and unreinforced) with membrane action induced by restraint against lateral displacement at the boundaries. The experimental part of the programme involved the making up and testing of 45 concrete slabs. In addition, the results of 22 slabs tested by other investigators are analysed. The investigation is divided into four main parts.

Firstly, a yield-line theory incorporating a rigid-plastic strip approximation is developed which defines the load-deflection curve at and after the ultimate flexural load for slabs with rigid boundaries during the compressive membrane action stage. Cases of boundary restraints which produce compressive membrane action in either one or two directions are considered and the accuracy of approximate yield-line patterns are investigated. On the basis of an empirical value, first for the width of the yield bands and thence for the central deflection at the ultimate load, expressions are obtained for the ultimate flexural strength of slabs which compare well with results obtained from slabs tested under short-term loading in very stiff surrounding frames.

Secondly, the theory is generalized to take into account the effects of axial strains in the slab and of small lateral
displacements at the boundaries, in order that the effects of long-term loading and elastic boundary restraints may be included. The theory is checked against test results obtained from slabs tested under periods of sustained loading. It is shown that axial strains and small lateral displacements at the boundaries can significantly reduce the compressive membrane action, especially in the case of thin slabs.

Thirdly, the behaviour of a slab as a tensile membrane at large deflections beyond that at the ultimate flexural load is examined. A theory is developed to define the load-deflection characteristics assuming that the reinforcement acts as a plastic membrane. It is shown that heavily reinforced slabs can carry loads by tensile membrane action which exceed the ultimate flexural load.

Finally, the laterally stiffness and strength required of surrounding beams and panels of slab and beam floors in order to enforce membrane action in interior panels is investigated. Results obtained from tests on the interior panels of 9 panel floors are analysed and show good agreement with the theory.

An appendix containing some results obtained by the conventional Johansen's yield-line theory is also included.
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Notation

\( x, y \) Rectangular coordinates in the plane (horizontal) of the slab.

\( z \) Vertical deflection at a plastic hinge of a strip, or, the vertical deflection of a yield line or region of yielding at point \((x, y)\) of a slab.

\( Y \) Transformed \( y \) axis for the equivalent simple tensile membrane.

\( a, b \) Half-lengths of the sides of the equivalent simple tensile membrane.

\( \Delta \) Maximum value of \( z \) for a slab at a particular load.

\( \Delta_u \) Maximum value of \( z \) for a slab at ultimate flexural load.

\( A_{st}, A_{sc} \) Cross-sectional areas, per unit width, of tension and compression steel, respectively.

\( A_t \) Area of tie steel in a supporting beam.

\( f'_s \) Yield stress of steel reinforcement.

\( f_s \) Stress in steel reinforcement.

\( f_{sc} \) Stress in the compression steel.

\( u \) Crushing strength of a 6in. or 4in. concrete cube.

\( k_1, k_3 \) The ratio of the mean compressive stress in the concrete of a section at ultimate load to the cube strength.

\( k_2 \) The ratio of the depth to the centroid of the compressive stress block at ultimate load to the
depth to the neutral axis, both distances being measured from the compressed face.

$k$ Factor depending upon the support condition at an edge.

$e_c$ Strain in concrete at the compressed edge when the section has reached its ultimate value.

$\varepsilon$, $\varepsilon_c$, $\varepsilon_s$ Longitudinal elastic, creep and shrinkage strains at mid-depth of a strip of a slab, respectively.

$\varepsilon$ Sum of longitudinal elastic, creep and shrinkage strains at mid-depth of a strip of a slab.

$E_s, E_c$ Young's modulus for steel and concrete, respectively.

$R$ Reduction coefficient to allow for axial strains and lateral displacements of the boundaries of a slab.

$K$ Factor depending upon the extent of cracking of concrete at a sustained load.

$W_u$ Uniformly distributed ultimate flexural load, per unit area, in the theory which includes membrane stresses.

$W$ Uniformly distributed load, per unit area, carried by the slab after the ultimate flexural load has been reached in the theory which includes membrane stresses.

$W_s$ Sustained uniformly distributed load, per unit area.
\( w_j \) Ultimate uniformly distributed load, per unit area, given by Johansen's yield-line theory.

\( \delta \) Virtual displacement of plastic hinge or yield line in the z direction.

\( \delta(x,y) \) Virtual displacement of point \((x,y)\) of slab in the z direction.

\( \theta \) Virtual rotation of a portion of a strip or a slab about its support edge.

\( L \) Span of a strip or a slab.

\( pL \) Length of the end portion of a strip.

\( L_1, L_2, L_3 \) Dimensions defining positions of yield lines of slabs.

\( L_t \) Length of tie steel in supporting beams.

\( L_p, L'_p \) Width of sagging and hogging moment yield bands, respectively.

\( t \) Outward movement of a slab edge at the boundary.

\( d \) Overall depth of strip or slab section.

\( n_1d, n'_1d \) Distances to neutral axes from the compressed faces of the concrete at sections with sagging and hogging bending moments, respectively.

\( d_1, d'_1 \) Distances from the compressed faces of the concrete to the centroids of the tension steel at sections with sagging and hogging bending moments, respectively.
\( d_2, d_2' \)  Distances from the compressed faces of the concrete to the centroids of the compression steel at sections with sagging and hogging bending moments, respectively.

\( C_c, C_c' \)  Resultant compressive forces in concrete, per unit width, at sections which have reached ultimate value with sagging and hogging bending moments, respectively.

\( C_s, C_s' \)  Resultant forces in the compression steel, per unit width, at sections with sagging and hogging bending moments, respectively.

\( T, T' \)  Resultant forces in the tension steel, per unit width, when at yield stresses at sections with sagging and hogging bending moments, respectively.

\( F \)  Tensile force, per unit width, of steel which crosses the whole span when at yield stress.

\( N, N' \)  Compressive membrane forces, per unit width, acting at sections with sagging and hogging bending moments, respectively.

\( M, M' \)  Yield moments of resistance, per unit width, at sections with membrane forces acting with sagging and hogging bending moments, respectively.
\( \text{m, } \text{m}' \) Yield moments of resistance, per unit width, at sections without membrane forces acting with sagging and hogging bending moments, respectively.

\( \text{m}^* \) Combination of yield moments depending upon the support conditions.

\( P \) Tie force in supporting beams for compressive membrane action.

\( \alpha, \beta, \gamma \) Constants for a given slab defined by equations (2.20), (2.21), (2.22), (2.24) and (2.25).

\( \varepsilon' \) Term including axial strain and lateral boundary displacement defined by equations (3.19) and (3.20).

Note: Further subscripts \( x \) and \( y \) denote quantities in the \( x \) and \( y \) directions, or quantities for \( x \) and \( y \) direction steel.
CHAPTER 1

INTRODUCTION AND SCOPE OF RESEARCH

1.1 Introduction

Continuous reinforced concrete slab and beam floors, consisting of two-way slabs or panels cast monolithic with supporting beams, are extensively used in structures. Traditionally, two-way slabs have been designed by either elastic theory or, more commonly, by use of approximate moment coefficients which are listed in codes of practice. More recently, the British code of practice for reinforced concrete buildings has also allowed a load-factor method of design to be used which recognises the redistribution of bending moments which occurs due to plasticity as ultimate load is approached. The deflection at working load is controlled by specifying maximum allowable values of the span/depth ratio. The load-factor method of design is becoming popular. The theory involved is relatively simple and exact solutions can be readily obtained for slabs with complex shapes, boundary conditions and loading, which could only be analysed with extreme difficulty by elastic theory. Simpler reinforcement layouts may be used and designs more economical than those provided by the approximate moment coefficient method can often be obtained.

1. All references may be seen listed at the end of the thesis.
The commonly used load-factor method, the "yield-line theory" or "fracture-line theory" due mainly to Johansen (see for example reference 2 and Appendix A), considers flexure only and neglects membrane action when determining the ultimate strength of slabs. Hence in the case of slabs with boundary conditions which allow collapse mechanisms to form in which membrane action is insignificant a good indication of the ultimate strength is obtained. Tests by Ockleston\textsuperscript{3,4}, Powell\textsuperscript{5}, Wood\textsuperscript{6} and the University of Illinois\textsuperscript{7}, however, have shown that two-way slabs which are restrained against lateral movement at the boundaries have ultimate loads which are far in excess of those predicted by Johansen's yield-line theory due to the development of compressive membrane action in the slab. This comes about from the increase in the ultimate flexural capacity of the slab at the yield lines due to the presence of compressive direct force as well as bending moment. This action has sometimes been termed "arching". The collapse mechanism formed at ultimate load, however, is of the conventional yield-line theory type, being composed of plane segments joined by lines of yielding.

The tests by Powell\textsuperscript{5} and Wood\textsuperscript{6} were conducted on uniformly loaded single panel units in which lateral displacements at the boundaries were prevented by massive
surrounding frames. These panels showed ultimate loads in the range 1.6 to 11.2 times the loads calculated by Johansen's yield-line theory. Ockleston\textsuperscript{3,4}, however, tested three interior panels of a full scale slab and beam building floor, and his results showed that the edge restraint provided by the stiffness of the surrounding panels was sufficient to cause the panels to fail at 2.6 to 2.9 times the Johansen loads. The University of Illinois\textsuperscript{7} test was of a 9 panel (3 by 3) slab and beam floor and the supporting beams failed when the load on the interior panel was at approximately twice the Johansen load.

If this enhanced load carrying capacity of interior panels of slab and beam floors is available it would seem that the design of such panels by yield-line theory neglecting the membrane action induced by the stiffness of the surrounding panels is very conservative. At present only extremely approximate theory exists to enable this enhanced strength to be determined. In this dissertation the effect of membrane action on the ultimate strength of uniformly loaded rectangular panels with restraint against lateral displacement at the boundaries will be considered.
1.2 Review of Previous Research into the Effect of Membrane Action on the Ultimate Strength of Concrete Slabs.

The effect of membrane action induced by composite action in continuous slab and beam floors has received surprisingly little attention in the past. Far more research has been conducted into other aspects of composite construction, such as the influence of walls on the stiffness of columns and supporting beams, and the well known T-beam action caused by slab and beam interaction.

One of the first records of measured ultimate loads being higher than those calculated by yield-line theory is found in a report by Thomas. This report commented upon the enhanced ultimate strength of beams and slabs which were tested with boundaries restrained against lateral movement. The effect of this type of boundary condition was probably made most widely known, however, as a result of the full scale loading tests carried out by Ockleston on a conventional reinforced concrete building. Three interior panels, each of dimensions 15.9ft. x 13.5ft. x 5.3in. and lightly reinforced, were uniformly loaded to failure. Ockleston showed that the enhanced ultimate strengths obtained (2.6 to 2.9 times the Johansen load) could not be attributed to the tensile strength of the concrete, nor to strain hardening or tensile membrane action of the reinforcement.
It was concluded that "arching forces" induced by the lateral restraint of the surrounding panels were responsible for the high ultimate loads. On the basis of the collapse mechanism given by simple yield-line theory and the measured deflection of the slab at ultimate load, and assuming that the compressive membrane forces were constant in all directions, Ockleston showed that membrane action could explain the high ultimate loads if the neutral axis depth at yield lines was approximately 0.5in. greater than that for a simple bending failure.

The University of Illinois have reported on the testing of a ¼ scale model of a 9 panel slab and beam floor. Each panel was 5ft. square and 1.5in. thick, and the floor was 3 panels wide in each direction. The loading applied was uniformly distributed. No attempt was made to derive a theory to explain the enhanced ultimate strength of the interior panel (the loading applied was almost twice the Johansen load when the supporting beams failed), other than to show approximately that tensile membrane action and strain hardening of the reinforcement could be discounted and to comment that at high deflections the increased bending moment at yield sections due to an apparent increase in the lever arm could account for the high ultimate load.
In the single panel tests conducted by Powell, slabs of 1.286in. thickness were clamped onto a very stiff surrounding steel frame to give spans of 36in. by 20.57in. The panels were loaded uniformly to failure and it was shown, experimentally, that compressive membrane action induced by restraint against lateral spread at the boundaries caused enhanced ultimate loads. The tests also showed that at large deflections after ultimate load the slab commenced to carry load by tensile membrane action. Powell made no attempt to produce a theory to explain his test results for slabs but commenced a theory for reinforced concrete beams with ends rigidly clamped against all movement. Assuming the material of the beam to be rigid-plastic and considering the requirements for compatible deformations of the collapse mechanism, the depths to the neutral axes at the yield sections were determined and a load-deflection relationship established which defined for beams with equal top and bottom steel the descending region of the load-deflection curve after ultimate load. This theory could be used to find the ultimate load of such beams if a method for determining the deflection of the beam at ultimate load was available.

Wood has reported on single panel tests conducted on uniformly loaded 68in. square slabs of 2\( \frac{1}{2} \)in. thickness with
boundaries restrained against all movement by a massive reinforced concrete surrounding frame. Load-deflection curves and enhanced ultimate loads similar to Powell were obtained. Using plate theory and assuming a rigid-plastic material Wood analysed the case of a uniformly loaded circular slab with edges fixed against all movement and obtained an equation which defined the descending region of the load-deflection curve after ultimate load. The load carried by a circular slab at high deflections after ultimate flexural load when the reinforcement acts as a tensile membrane was also considered. For the more practical case of a rectangular slab an exact solution was not attempted, but a semi-empirical method for determining the ultimate load was investigated. This method was as follows. The maximum possible ultimate moment of resistance which a section subjected to combined bending and direct compressive force can attain was calculated. This bending moment was substituted into the Johansen's yield-line theory expression for ultimate load in the place of the simple ultimate moment of resistance which neglects membrane force. Thus the maximum possible ultimate flexural strength of a slab was obtained. Then recognizing that the actual ultimate moments at the yield lines lie somewhere between the magnitudes of
the maximum possible ultimate moment with membrane action
and the simple ultimate moment without membrane action, the
experimental results of his own slabs and those of
Ockleston and Powell were examined to find the ratio of
experimental ultimate load to maximum possible theoretical
ultimate load. These ratios varied with the steel contents
of the slabs but showed a trend and could be used to estimate
the actual ultimate loads of rectangular slabs from the
maximum possible theoretical ultimate loads.

More recently Christiansen\textsuperscript{9} has developed a theory for
beams and one-way slabs with the supported edges restrained
against lateral displacement. The theory includes the
effects of shortening of the beam due to elastic deformations
and the outward movement of supports due to elastic restraints.
By considering the requirements of the compatibility of
deformations an expression was obtained for the load carried
by "arching". By finding the deflection at which the load
carried by arching is a maximum the ultimate load of the
beam was determined. Christiansen, however, considers the
plastic hinges to be fully developed at all stages in his
analyses and it is apparent that the ultimate load given by his
theory could be supposed to occur at deflections which are too
small to develop full plasticity at the hinge positions.
Christiansen did not extend his theory to two-way slabs other
than to comment that the theory could be used to calculate
the arching across the middle of the short span.

The above review of existing research shows that there is conclusive experimental evidence of the beneficial effect of membrane action on the ultimate strength of slabs, but that at present theory has not been developed to fully explain the various characteristics of slabs with membrane action.

1.3 The Load-Deflection Curve of Uniformly Loaded Laterally Restrained Slabs.

The tests conducted by Powell and Wood show that the effect of restraint against lateral displacement at the edges of uniformly loaded two-way concrete slabs is to produce load versus central deflection curves of the form shown in Fig.1.1. The curves may be divided into three distinct regions:

(a) Region A to B. This is the region of increasing compressive membrane action. The initial linearity of the curve is due to elastic deformation of the uncracked slab. The onset of cracking is delayed by the restraint against lateral displacement at the slab boundaries. In most cases the theoretical Johansen ultimate load is reached and sometimes well exceeded before cracking becomes visible.

As the load is increased beyond the stage of cracking the reinforcement and the compressed concrete commence to become
FIG.1.1 LOAD-DEFLECTION CURVES FOR LATERALLY RESTRAINED TWO-WAY CONCRETE SLABS.
plastic at the critical sections. With further loading the plasticity extends along well defined lines of intensive cracking and the slab is converted into a mechanism formed of almost plane segments connected by lines of plastic hinges. Aided by compressive membrane forces the ultimate flexural load of the slab is reached at point B with a central deflection of 0.33 to 0.5 of the slab thickness.

(b) Region B to C. This is the region of decreasing compressive membrane action. As the deflection of the slab is increased beyond that required to cause the ultimate flexural load the load carried by the slab decreases rapidly due to the compressive membrane forces becoming smaller. The increase in deflection of the slab in the early part of this region is due almost entirely to rotation of the slab segments about the yield lines. As point C is approached the membrane forces in the central region of the slab commence to change from compression to tension with cracks extending throughout the depth of the concrete due to the large stretch of the slab surface. Unreinforced slabs will carry no further load once this stage is reached. The central deflection at the point C is approximately equal to the thickness of the slab.

(c) Region C to D. This is the region of increasing tensile membrane action. Although once the point C is reached the concrete in the central region of the slab can carry no
further load due to cracks penetrating its full thickness, the stress and deflection of the reinforcement is great enough to cause load to be carried by tensile membrane action. With further deflection the load carried by tensile membrane action increases, and in heavily reinforced slabs yielding of steel and full depth cracking of the concrete spreads throughout the slab. The increase in load carried by the slab with further deflection stops only when the reinforcement commences to fracture. In heavily reinforced slabs the load carried by tensile membrane action at point D may exceed the ultimate flexural load which was carried at point B.

The load-deflection curves of Fig. 1.1 show that the load carried by the slab falls off suddenly once the ultimate flexural load has been reached. Also it should be noted that the load-deflection curves described above are only obtained if a hydraulic load system incorporating an almost incompressible fluid (for example conventional hydraulic rams or a pressure bag filled with water) is used which, due to fall off in pressure as the slab deflects, allows the descending region B to C of the load-deflection curve to be followed. If the load causing the failure of the slab is the practical case of gravity loading (for instance, superimposed dead
weights), which remain unchanged as the slab deflects, the load-deflection curve will run from A to B as before and then show a straight horizontal line between B and the intersection with the curve CD. Thus if the point D is not at a greater load than the point B gravity loading will drop suddenly through the slab causing a catastrophic failure. It should also be noted that the load-deflection curves described above are also given by beams and one-way slabs which are restrained against lateral displacement at the supported edges.

1.4 **Scope of Work to be Developed.**

The aspects of the ultimate strength of uniformly loaded two-way concrete slabs with laterally restrained edges investigated in the following chapters will be as follows:

(a) Theory will be developed to define the portion of the load-deflection curve between points B and C of Fig.1.1. The central deflection of the slab at point B will then be determined so that the ultimate flexural strength may be obtained. Rectangular slabs with all or three edges fully restrained will be considered, these being the ideal cases of interior panels and edge panels of continuous slab and beam floors.
(b) The effect of elastic, creep and shrinkage strains in the plane of the slab will be investigated since long-term loading effects may reduce compressive membrane action considerably. The reduction in ultimate flexural load caused by outward movement of the slab boundaries will also be considered.

(c) The load-carrying capacity of the slab as a tensile membrane at very large deflections will be investigated.

(d) The stiffness and strength of surrounding panels required to enforce membrane action in a loaded panel of a slab and beam floor will be considered.

In addition, an appendix containing some results obtained by conventional Johansen's yield-line theory will be included for comparison with the results obtained from the theory which includes membrane stresses.
CHAPTER 2

THE ULTIMATE FLEXURAL STRENGTH OF RECTANGULAR TWO-WAY CONCRETE SLABS UNDER SHORT-TERM UNIFORM LOADING WITH ALL OR SOME EDGES LATERALLY RESTRAINED: RIGID-PLASTIC RIGID-BOUNDARY THEORY.

2.1 Scope of Theory and Tests

Theory will be developed firstly to define the descending portion of the load-deflection curve (the curve BC of Fig.1.1) and then the central deflection at the ultimate flexural load will be determined in order that the ultimate flexural load may be ascertained. Two cases of boundary conditions will be considered. The first case is that of a slab with all edges completely restrained against movement. This may be regarded as the ideal case of an interior panel of a continuous slab and beam floor. The second case is that of a slab with one edge free to translate horizontally but with the remaining edges completely restrained. This may be regarded as the ideal case of either an edge panel of a continuous slab and beam floor, or of an interior panel which has a narrow edge panel on one side.

The theory developed will be compared with the test results obtained by Powell and Wood, and also against further test results obtained as a part of this investigation.
2.2 Theory For Load Carried At and After Ultimate Flexural Load With Compressive Membrane Forces Acting.

2.2.1 Ultimate Conditions At a Yield-Line.

Slabs which have edges parallel to the x and y axes and which are reinforced by steel placed in the x and y directions will be considered. The cross-sectional area of each layer of reinforcement, per unit width, will be considered to be constant across the slab, but may be different for x and y direction reinforcement and for top and bottom reinforcement. The sagging moment (bottom) steel will be considered to be placed over the whole of the slab, but the hogging moment (top) steel need only extend far enough into the slab from the supports to ensure that the hogging moment yield lines form at the supports. In order to develop expressions for conditions at yield lines it will be assumed that plane sections before bending remain plane after bending and that concrete has no tensile strength. These two assumptions are usually made in reinforced concrete theory.

Consider a section on a yield line. Let the steel run normal to the section and let the compressed concrete be at its ultimate strength under the action of a sagging bending moment $M$, per unit width, and a compressive membrane force $N$, per unit width. Since for the slabs considered the strain in the concrete at the compressed edge will be in the
order of 0.003 to 0.004 and the depth to neutral axis will always be less than one half of the slab thickness, the tension steel at the section will be at yield stress but the compression steel may not be yielding. Reference to Fig.2.1 shows:

Compressive force in the concrete, per unit width,
\[ C_c = k_1 k_3 u_n_1 d. \]
Force in the compression steel, per unit width,
\[ C_s = A_s f_s c. \]
Force in the tension steel, per unit width,
\[ T = A_t f_t'. \]
Then the compressive membrane force, per unit width, is given by:
\[ N = k_1 k_3 u_n_1 d + C_s - T. \] (2.1)

And the moment of resistance, per unit width, is given by:
\[ M = k_1 k_3 u_n_1 d (0.5d - k_2 n_1 d) + C_s (0.5d - d_2) + T (d_1 - 0.5d). \] (2.2)

Similarly expressions for \( N' \) and \( M' \) for sections with hogging bending moment may be written:
\[ N' = k_1 k_3 u_n_1 d + C_s' - T'. \] (2.3)
\[ M' = k_1 k_3 u_n_1 d (0.5d - k_2 n_1 d) + C_s' (0.5d - d_2) + T' (d_1 - 0.5d). \] (2.4)
FIG. 2.1 CONDITIONS AT A SECTION ON A YIELD LINE
The depths to the neutral axes, defined by \( n_1 \) and \( n_1' \), are dependent upon the deflection of the slab and will be determined later. The stress in the compression steel may be found by using the similar triangles of the strain diagram of Fig.2.1:

\[
\text{If } e_c \left( \frac{n_1 d - d_2}{n_1 d} \right) > \frac{f'_s}{E_s}, \text{ then } f_{sc} = f'_s.
\]

\[
\text{And if } e_c \left( \frac{n_1 d - d_2}{n_1 d} \right) < \frac{f'_s}{E_s}, \text{ then } f_{sc} = e_c E_s \left( \frac{n_1 d - d_2}{n_1 d} \right).
\]

In order to apply (2.1), (2.2), (2.3) and (2.4) to a given section the values of the strain \( e_c \) and of the coefficients \( k_1, k_2, \) and \( k_2 \) are required. The values of the coefficients, in particular, are required accurately as the strength of the slab is very much dependent upon the compressive membrane force. A large amount of research has been conducted on concrete in the past to determine these properties and differing results have been obtained. Probably the most comprehensive series of tests, the results of which are becoming a standard reference, were those conducted by Hognestad, Hanson and McHendry\(^{10}\). Their results as published are related to the crushing strength of a 6in. diameter x 12in. cylinder. To relate them to the crushing strength of a 6in. cube, the standard British test specimen, the cylinder strength will be taken to be 0.8 of the cube strength.
This is a commonly used conversion factor. Fig.2.2 shows the values of $k_1, k_3, k_2$ and $e_c$ plotted for various cube strengths. These values will be used in this thesis. It is to be noted that they were obtained from tests on uniaxially loaded specimens and are therefore probably conservative for a slab under biaxial compression.

2.2.2 The Rigid-Plastic Strip With Ends Fixed Against Rotation and All Translation.

The theory for slabs will be commenced by discussing the case of a loaded strip of unit width with its ends fixed against rotation and all translation and its long edges free. Let the strip be loaded up to and beyond ultimate load so that the central deflection is in the region B to C of Fig.1.1. Although the ultimate load of the strip is much higher than that calculated by simple plastic hinge theory, due to the compressive membrane forces induced by the restraint against spread, the collapse mechanism which forms at B of Fig.1.1 is similar to that expected from simple plastic hinge theory since the membrane force is constant along the length of the strip. Fig.2.3 shows the general collapse mechanism of a strip with equal amounts of top steel at each support. Powell has analysed the case of a strip with equal amounts of tension steel at mid-span and supports and no compression steel. The theory will now be extended to the case of a
FIG. 2.2 PROPERTIES OF COMPRESSIVE STRESS BLOCK RELATED TO CUBE STRENGTH.
FIG. 2.3 COLLAPSE MECHANISM OF STRIP WITH FULLY FIXED ENDS.

FIG. 2.4 INTERNAL ACTIONS AT YIELD SECTIONS OF END PORTION OF STRIP.
strip containing both tension and compression steel and with different amounts of steel at mid-span and supports. To analyse this case assume:

(a) The material of the strip is "rigid-plastic". Hence the elastic shortening of the strip due to axial force is neglected and the portions of the strip between the yield sections remain straight. Also, since the theory is for short-term loading, creep and shrinkage strains are assumed to be zero.

(b) The stresses at the yield sections (i.e. the plastic hinges) retain their yield values. Hence it is assumed that the steel shows no strain hardening and a reasonable amount of rotation can occur at the section after plasticity sets in with the concrete acting at its ultimate value.

(c) The deflection of the strip at the central yield sections is in the direction of the normal to the line joining the ends of the strip. This requires the external loading and the reinforcement of the strip to be symmetrical about mid-span.

On the basis of these assumptions the depth to the neutral axis at each yield section may be found by considering the compatibility of deformations. Fig.2.4 shows the portion of the strip between yield sections 1 and 2 of Fig.2.3.
Since no horizontal displacement of yield sections occurs, the horizontal distance between yield sections 1 and 2 remains $\rho L$ and the distance between points A and C of Fig. 2.4 is:

$$\rho L \sec \phi = \rho L + AB - CD.$$  

$$= \rho L + (1-n')d \tan \phi - n'd \tan \phi.$$  

Hence,

$$(1-n' - n_1) = \frac{\rho L(\sec \phi - 1)}{d \tan \phi} = \frac{2\rho L \sin^2 0.5\phi}{d \sin \phi}.$$  

But $\sin \phi = \frac{\zeta}{\rho L}$ and $\sin 0.5\phi = \frac{\zeta}{2\rho L}$, since $\phi$ is small.

$$\therefore 1 - n' - n_1 = \frac{\zeta}{2d}.$$  

Hence,

$$n'_1 + n_1 = 1 - \frac{\zeta}{2d}.$$  \hspace{1cm} (2.5)

Also, for equilibrium, the sum of the horizontal forces acting on portion 12 of the strip is zero. Hence:

$$C'_c + C'_s - T' = C'_c + C'_s - T, \text{ or } N' = N.$$  \hspace{1cm} (2.6)

$$\therefore k_1k_2ud(n'_1 - n_1) + C'_s - C'_s - T' + T = 0.$$
Hence:

\[ n'_1 - n_1 = \frac{T' - T - C'_S + C_S}{k_1 k_3 ud} \]  \hspace{1cm} (2.7)

On solving (2.5) and (2.7) simultaneously:

\[ n'_1 = 0.5\left(1 - \frac{z}{2d} + \frac{T' - T - C'_S + C_S}{k_1 k_3 ud}\right) \]  \hspace{1cm} (2.8)

\[ n_1 = 0.5\left(1 - \frac{z}{2d} - \frac{T' - T - C'_S + C_S}{k_1 k_3 ud}\right). \]  \hspace{1cm} (2.9)

Now from (2.6), \( N' = N \).

Hence substituting (2.9) into (2.1) gives:

\[ N' = N = 0.25k_1 k_3 u(2d - z) - 0.5(T' + T - C'_S + C_S). \]  \hspace{1cm} (2.10)

Also substituting (2.8) and (2.9) into (2.4) and (2.2) respectively, gives:

\[ M' = \frac{1}{16} k_1 k_3 u[4d^2 (1-k_2) + 2zd(2k_2 - 1) - z^2 k_2] \]

\[ + C'_S (0.5d - d'_2) + T'(d'_1 - 0.5d) + 0.25(T' - T - C'_S + C_S)x \]

\[ [d(1 - 2k_2) + k_2 z] - \frac{k_2}{4k_1 k_3 u} (T' - T - C'_S + C_S)^2 \]  \hspace{1cm} (2.11)
\[ M = \frac{1}{16} k_1 k_3 u [4d^2 (1-k_2) + 2zd(2k_2-1) - z^2k_2] + C_S (0.5d-d_2) \]

\[ + T(d_1-0.5d) + 0.25(T'-T-C'_S+C_S) [d(2k_2-1) - k_2 z] \]

\[ - \frac{k_2}{4k_1 k_3 u} (T'-T-C'_S+C_S)^2. \]  \hspace{1cm} (2.12)

It will be noticed that \( N', N, M' \) and \( M \) are all dependent upon \( z \), the vertical deflection of the central portion of the strip, but are independent of \( \rho L \), the length of the end portions of the strip. Now the vertical distance between the lines of action of \( N' \) and \( N \) is \( z \). Hence considering portion 12 of the strip, the sum of the internal forces at yield sections about one end is:

\[ M' + M - Nz, \text{ per unit width.} \]  \hspace{1cm} (2.13)

Shear forces have been neglected since their net contribution to the analysis by virtual work will be zero. If the portion 12 of strip is given a virtual rotation \( \Theta \) by giving the centre of the strip a virtual displacement in the direction of the loading, the virtual work done by the internal actions at yield sections is:

\[ -(M' + M - Nz) \Theta, \text{ per unit width.} \]  \hspace{1cm} (2.14)
By adding (2.14) to the virtual work done by the external loading and equating the result to zero an equation is obtained relating the central deflection \( z \) to the loading \( w \), thus defining the portion B to C of the curve of Fig.1.1 for the strip.

Also if \( N' = N = 0 \), the membrane stresses are zero and

\[
C' + C'_s - T' = 0, \text{ hence } n'_1 = \frac{T' - C'_s}{k_1 k_3 ud} \tag{2.15}
\]

\[
C + C_s - T = 0, \text{ hence } n_1 = \frac{T - C_s}{k_1 k_3 ud} \tag{2.16}
\]

Then the moments \( M' \) and \( M \) become the yield moments without membrane stresses, \( m' \) and \( m \), and from (2.4) and (2.15), and (2.2) and (2.16), respectively:

\[
m' = C'_s (d'_1 - d'_2) + (T' - C'_s) (d'_1 - \frac{k_2}{k_1 k_3} \frac{T' - C'_s}{u}) \tag{2.17}
\]

\[
m = C_s (d_1 - d_2) + (T - C_s) (d_1 - \frac{k_2}{k_1 k_3} \frac{T - C_s}{u}). \tag{2.18}
\]

Normally in (2.10), (2.11) and (2.12) the compression steel will be at yield stress, but in (2.17) and (2.18) in most cases the force in the compression steel can be neglected since the compression steel will be close to the neutral axis and will contribute little to the strength.
2.2.3 **The Strip Approximation For a Uniformly Loaded Slab With Edges Fully Restrained Against Rotation and All Translation.**

In order to analyse the case of a two-way slab a strip approximation will be used. In addition to the assumptions made in section 2.2.2 when developing expressions for strips, the following assumptions will be made:

(a) The slab is composed of strips running in the x and y directions which have the same depth as the slab. The strips in the x direction contain only the x direction steel and strips in the y direction contain only the y direction steel. The external loading on the slab is shared between the two systems of strips and the vertical deflections are the same as for the slab. A slab which is considered to be made up of strips as above will be referred to as the "equivalent slab".

(b) The configuration of the yield lines of the collapse mechanism of the equivalent slab is of the same type as for the identical slab in which the edges are not restrained against horizontal displacement except that each corner "fan" is replaced by a single straight yield line, e.g. see Fig.2.5. Membrane forces, however, may cause the values of the angles between the yield lines to be different from those for the slab in which membrane
forces do not act. That this general type of yield-line pattern forms has been confirmed experimentally. Theoretically, corner fans do form, however, and replacing them by single yield lines will cause the ultimate strength of the slab to be slightly overestimated, since an upper bound approach is used.

(c) The tension steel has yielded and the concrete has reached its maximum compressive value at the sections of the strips which lie on the yield lines.

(d) The yield sections of the strips are at right angles to the directions of the steel and the torsional moments acting on these sections are zero. This follows conventional yield-line theory which considers a yield line to be "stepped" in the directions of the steel and assumes the torsional moments acting on the faces of the steps to be zero.

Each strip of the equivalent slab when yielding at the sections indicated by the yield-line pattern is divided into 2 or 3 straight portions separated by the yield sections. For example see Fig.2.5. On the basis of the above assumptions it is evident that each end portion of a strip is behaving similarly to the portion 12 of Fig.2.4, although of different lengths. Thus the
actions at the yield sections of the strips are given by (2.10), (2.11) and (2.12), written for the x and y direction steel of the slab, and it is to be noticed that these actions are independent of the lengths of the portions of the strips but are dependent on \( z \), the relative vertical deflections of the yield sections. \( z \) is variable over the slab.

Now substituting (2.10), (2.11) and (2.12) into (2.13) and writing the resulting expression for x and y direction strips gives:

\[
M'_x + M_x - N_x z = \alpha_x + \beta_x z + \gamma z^2, \tag{2.19}
\]

where:

\[
\alpha_x = 0.5k_1k_3(1-k_2)u d^2 - \frac{k_2}{2k_1k_3u} (T'_x-T_x-C'_sx+C_{sx})^2
+ C_{sx}(0.5d-d_x') + C'_{sx}(0.5d'-d_x') + T_x(d_x'-0.5d)
+ T'_x(d_x'-0.5d) \tag{2.20}
\]

\[
\beta_x = 0.25k_1k_3(2k_2-3)u d + 0.5(T'_x+T_x-C'_sx-C_{sx}) \tag{2.21}
\]

\[
\gamma = 0.125k_1k_3(2-k_2)u. \tag{2.22}
\]
And

\[ M'_y + M'_y - N'_y z = \alpha_y + \beta_y z + \gamma z^2, \quad (2.23) \]

where

\[
\alpha_y = 0.5k_1 k_3 (1-k_2) ud^2 - \frac{k_2}{2k_1 k_3 u} (T'_{y} - T_{y} - C'_{sy} + C_{sy})^2 \\
+ C_{sy} (0.5d - d'_{y}) + C'_{sy} (0.5d - d'_{y}) + T_{y} (d'_{y} - 0.5d) \\
+ T'_{y} (d'_{y} - 0.5d) \quad (2.24)
\]

\[
\beta_y = 0.25k_1 k_3 (2k_2 - 3) ud + 0.5(T'_{y} + T_{y} - C'_{sy} - C_{sy}) \quad (2.25)
\]

\[ \gamma \text{ is as in } (2.22). \]

Note that \( \alpha_x, \alpha_y, \beta_x, \beta_y \) and \( \gamma \) are constants for a given slab.

Also, writing (2.17) and (2.18) for \( x \) and \( y \) directions strips and neglecting the force in the compression steel since when membrane forces are zero the compression steel will be close to the neutral axis:

\[
m'_{x} = T'_{x} (d'_{x} - \frac{k_2 T'_{x}}{k_1 k_3 u}) \quad (2.26)
\]

\[
m_{x} = T_{x} (d_{x} - \frac{k_2 T_{x}}{k_1 k_3 u}) \quad (2.27)
\]
\[ m_y' = T_y' (d_y' - \frac{k_2 T_y}{k_1 k_3 u}) \]  

(2.28)

\[ m_y = T_y (d_y - \frac{k_2 T_y}{k_1 k_3 u}). \]  

(2.29)

To find the relationship between external load and central deflection for the region B to C of Fig.1.1 using the principle of virtual work the equivalent slab is given a small virtual displacement and the equation of virtual work is written. If the virtual displacement at any point \((x,y)\) on the slab is \(\delta(x,y)\) in the direction of the load and this results in virtual rotations \(\theta_x\) and \(\theta_y\) of the end portions of \(x\) and \(y\) direction strips respectively (the centre portions of those strips divided into three portions will not undergo rotation), then:

Virtual work done by the external load, \(w\) per unit area,

\[ = \int \int w \delta(x,y) \, dx \, dy. \]  

(2.30)

Virtual work done by \(x\) direction strips, from (2.14) and (2.19)

\[ = -\Sigma f(\alpha_x + \beta_x z + yz^2) \theta_x \, dy. \]  

(2.31)

(Integrated over the whole width of each set of strips and the contributions from each set of strips summed).
Virtual work done by y direction strips, from (2.14) and (2.23)

\[ \frac{\partial w}{\partial L_1} = \frac{\partial w}{\partial L_2} = \ldots = 0. \]  

(2.34)

Hence the virtual work equation is:

\[ 0 = \iint w \delta(x,y) \, dx \, dy - \int \int (\alpha_x + \beta_x z + \gamma z^2) \, Q_x \, dy \\
- \int \int (\alpha_y + \beta_y z + \gamma z^2) \, Q_y \, dx. \]  

(2.33)

It is to be noted that if the membrane stresses in one direction are zero, then \( N = 0, M' = m' \) and \( M = m \) are substituted into (2.33) for the direction concerned.

The relationship between the external load and the central deflection of the slab is given by (2.33) in terms of the constants \( \alpha, \beta \) and \( \gamma \) defining the properties of the slab, the slab dimensions and variables \( L_1, L_2, \ldots \) etc. which define the positions of the yield lines. The values of \( L_1, L_2, \ldots \) etc. which make the external load a minimum at a given deflection are found by solving simultaneously the equations
It is to be noted that although (2.33) and (2.34) give a relationship between the external load and the central deflection which defined the curve BC of Fig.1.1 for the slab, no indication of the range over which it is applicable is obtained and to determine the points B and C on the curve some other means must be sought.

2.2.4 Uniformly Loaded Rectangular Two-Way Slab With All Edges Fully Restrained Against Rotation and Translation.

Consider a slab with all edges completely restrained against movement. Let the slab be loaded beyond the point of ultimate load so that the maximum deflection \( \Delta \) lies somewhere between points B and C of Fig.1.1. Then the yield-line pattern and the yield sections of the strips which make up the equivalent slab will be as shown in Fig.2.5 for slabs with \( L_y > L_x \). The positions of the yield sections are defined by the dimension \( L_2 \), and due to symmetry the line of yielding EF lies at the centre of the slab.

The origin of coordinates is at A and \( z \) is the vertical deflection at any point with coordinates \((x,y)\) at the yield sections of the strips. Since \( z = \Delta \) at line EF, at any point with coordinates \((x,y)\) on yield line AE:

\[
z = \frac{2\Delta x}{L_x} = \frac{\Delta y}{L_2}.
\]
YIELD-LINE PATTERN OF ACTUAL SLAB

STRIPS OF EQUIVALENT SLAB

FIG. 2.5 UNIFORMLY LOADED TWO-WAY SLAB WITH ALL EDGES FULLY FIXED
The deflection at points on the other yield lines DE, BF, and CF follow from the above by symmetry. Now since \( z \) is a variable, the actions at the yield sections of \( x \) and \( y \) direction strips, \( M_x', M_y', M_x, M_y, N_x \) and \( N_y \), will also be variables depending upon the position \((x,y)\) of the yield section under consideration.

Suppose that the equivalent slab is given a virtual displacement \( \delta \) at the yield line EF in the direction of the loading. The slab segments, made up from portions of strips and bounded by lines of yielding, will undergo virtual rotations about the lines of yielding at the edges of the slab.

Virtual rotations of segments ABFE and DCFE about AB and CD respectively
\[
= \frac{2\delta}{L_x}.
\]

Virtual rotations of segments ADE and BCF about AD and BC respectively
\[
= \frac{\delta}{L_y}.
\]

Let the strips in the \( x \) and \( y \) directions be \( dy \) and \( dx \) in width, respectively.

Then the virtual work done by internal actions at the yield sections of the system of strips in the \( x \) direction =
Virtual work done by end portions of strips in segments ABFE and DCFE only (since internal actions in x direction strips in segments ADE and BCF have no corresponding displacement), and from (2.31)

\[ -4 \int_0^{L_x} \left( \alpha_x + \beta_x z + yz^2 \right) \frac{2 \delta}{L_x} \, dy - 2 \left( \alpha_x + \beta_x z + yz^2 \right) \frac{2 \delta}{L_x} (L_y - 2L_a) \]

\[ = - \frac{8 \delta}{L_x} \int_0^{L_x} (\alpha_x + \beta_x \frac{\Delta y}{L_a} + \frac{\Delta^2 y^2}{L_a^2}) \, dy - \frac{4 \delta}{L_x} (L_y - 2L_a) (\alpha_x + \beta_x \Delta + \gamma \Delta^2) \]

\[ = -4 \delta \left\{ \frac{L_y}{L_x} \alpha_x + \Delta \left( \frac{L_y - L_a}{L_x} \right) \beta_x + \Delta^2 \left( \frac{3L_y - 4L_a}{3L_x} \right) \gamma \right\} \]  \hspace{1cm} (2.35)

Similarly, the virtual work done by internal actions at the yield sections of the system of strips in the y direction = Virtual work done by end portions of strips in segments ADE and BCF only, and from (2.32)

\[ -4 \int_0^{0.5L_y} \left( \alpha_y + \beta_y z + yz^2 \right) \frac{5 \delta}{L_a} \, dx \]

\[ = - \frac{4 \delta}{L_a} \int_0^{0.5L_y} \left( \alpha_y + \beta_y \frac{2 \Delta x}{L_x} + \gamma \frac{4 \Delta^2 x^2}{L_x^2} \right) \, dx \]

\[ = - \delta \left\{ \frac{2L_y}{L_a} \alpha_y + \Delta \frac{L_y}{L_a} \beta_y + \Delta^2 \frac{2}{3} \frac{L_y}{L_a} \gamma \right\} \]  \hspace{1cm} (2.36)

Also virtual work done by the external loading of w per unit area, from (2.30)
\[ w = w \{ 2L_x L_a \frac{\Delta}{2} + (L_y - 2L_a) L_x \frac{\Delta}{2} \} \]

\[ = \frac{w \Delta L_x}{6} (3L_y - 2L_a). \quad (2.37) \]

Now from the equation of virtual work, (2.33),

\[ (2.37) + (2.35) + (2.36) = 0. \]

From which \( \Delta \) cancels and the following expression for \( w \) is determined for when \( L_y > L_x \):

\[ w = \frac{6}{L_x (3L_y - 2L_a)} \left\{ 4 \frac{L_y}{L_x} \alpha_x + 2 \frac{L_x}{L_a} \alpha_y + \Delta \left[ 2 \frac{L_y}{L_x} \beta_x + \frac{L_x}{L_a} \beta_y \right] \right. \]

\[ + \Delta^2 \left[ \frac{4}{3} \frac{(3L_y - 4L_a)}{L_x} + \frac{2L_x}{3L_a} \right] \}

\[ (2.38) \]

The value of \( L_a \) required for (2.38) is that value which gives minimum \( w \) at a given \( \Delta \), and such a value is found from \( \frac{\partial w}{\partial L_a} = 0 \).

The result of this operation is to give \( L_a \) as the solution of the following equation:

\[ 0 = \left( \frac{L_a}{L_x} \right)^2 \frac{4L_y}{L_x} (2\alpha_x - 3\beta_x - 2\Delta^2 \gamma) + \left( \frac{4L_a}{3L_x} - \frac{L_y}{L_x} \right) (6\alpha_y + 3\beta_y + 2\Delta^2 \gamma). \quad (2.39) \]

It is evident that when the value of \( L_a \) from (2.39) is
substituted into (2.38) the relationship between \( w \) and \( \Delta \) obtained is lengthy and some means of simplifications is desirable. Now a useful property of (2.38) is that if a value of \( \Lambda_2 \) which differs by a small amount from the exact value of \( \Lambda_2 \) (given by (2.39)) is substituted into it, a value of \( w \) is obtained which differs only slightly from the true minimum value. It is shown in Appendix A that in the case of Johansen's yield-line theory the assumption of \( \Lambda_2 = 0.5L_x \) leads to an error in the value of \( w \) found using the virtual work method which is not greater than 3% for uniformly loaded rectangular slabs with edges fixed against rotation and containing practical amounts of reinforcement.

The use of this simplification for the case including membrane stresses will now be examined. If \( \Lambda_2 = 0.5L_x \) is substituted into (2.38) and the resulting \( w \) is compared with the exact value of \( w \) given by (2.38) and (2.39), it can be shown that the error in \( w \) is also not more than 3% for the case of slabs with \( \frac{L_y}{L_x} \) ratios between 1 and 3, cube strengths of 3,000 lb./sq.in. and mild steel reinforcement contents of 0.15% of the concrete section in each direction (the minimum steel content allowed by C.P. 114: 19571), at a central deflection of one half of the slab thickness. Slabs with minimum steel were checked, since then the effect of
membrane stresses is greatest, and a central deflection of one half of the slab thickness was used since it has been shown experimentally\textsuperscript{5,6} that this is the approximate deflection at the point B of Fig.1.1. For more heavily reinforced slabs and different concrete and steel strengths the error in \( w \) will be not much different from the above value. Thus the approximation of \( L_2 = 0.5L_x \) gives a good value for \( w \) (the approximate yield-line pattern is shown in Fig.2.8), and for slabs with \( L_y > L_x \) (2.38) may then be written as:

\[
w = \frac{2L}{L_x(3L_y/L_x - 1)} \left\{ \frac{L_y}{L_x} \alpha_x + \alpha_y + \Delta \left[ \frac{L_y}{L_x} - \frac{1}{2} \right] \beta_x + \frac{1}{2} \beta_y \right\} \\
+ \Delta^2 \left( \frac{L_y}{L_x} - \frac{1}{2} \right) \gamma
\] 

(2.40)

Where \( \alpha_x, \alpha_y, \beta_x, \beta_y \) and \( \gamma \) are given by (2.20), (2.24), (2.21), (2.25) and (2.22).

In the above analysis only the yield-line pattern shown in Fig.2.5 has been considered. There is an alternative type of yield-line pattern possible which has the centre sagging moment yield line running parallel to the short edges of the slab. It may be shown, however, that this alternative pattern will not develop for slabs with all edges fixed.
2.2.5 Uniformly Loaded Rectangular Two-Way Slab With Three Edges Fully Restrained Against Rotation and Translation and the Remaining Edge Free to Translate Horizontally.

(i) The effect of the unrestrained edge: The presence of an edge which is not restrained against horizontal displacement will mean, according to the strip approximation, that membrane forces will not develop in the direction of the span at right angles to that edge. Due to the unsymmetrical boundary conditions and to membrane forces acting in one direction only there are two possible yield-line patterns for each of the cases of a long edge or a short edge unrestrained. The minimum load given by each alternative yield-line pattern will now be derived and the critical yield-line pattern found by determining which gives the absolute minimum load. In the analysis to follow, the edge free to translate horizontally may be either simply supported or fixed against rotation.

(ii) A long edge unrestrained laterally: Let the slab be loaded beyond the point of ultimate load so that the maximum deflection $\Delta$ lies somewhere between the points B and C of Fig.1.1. The alternative yield-line patterns of the equivalent slab are shown in Fig.2.6 for the case when the unrestrained edge is simply supported. If the unrestrained
FIG. 2.6 UNIFORMLY LOADED TWO-WAY SLAB WITH A LONG EDGE SIMPLY SUPPORTED AND REMAINING EDGES FULLY FIXED

FIG. 2.7 UNIFORMLY LOADED TWO WAY SLAB WITH A SHORT EDGE SIMPLY SUPPORTED AND REMAINING EDGES FULLY FIXED
edge is fixed against rotation (but not against horizontal displacement) the yield-line patterns will be as in Fig. 2.6 with the addition of a hogging moment yield-line along the laterally unrestrained edge.

a) Alternative yield-line pattern 1:

The yield-line pattern is shown in Fig. 2.6a. The positions of the yield lines are defined by dimensions $L_1$ and $L_2$. The dimensions marked $L_2$ are equal due to symmetry.

The origin of coordinates is at $A$. If $z$ is the vertical deflection at any point with coordinates $(x,y)$ at the yield sections of the strips, and $z = \Delta$ at the line $EF$; then:

$$z = \frac{x}{L_1} \Delta = \frac{y}{L_2} \Delta \text{ at any point on line } AE$$

$$z = \frac{L_2 - x}{(L_2 - L_1)} \Delta = \frac{y}{L_2} \Delta \text{ at any point on line } DE.$$  

The deflection at points on lines $BF$ and $CF$ follow from above by symmetry.

Since the edge $AB$ is free to move laterally the actions at the yield sections of the $x$ direction strips are written noting that there is no membrane force in the $x$ direction ($N_x = 0$), and that therefore the actions are the Johansen
yield moments \(m_x'\) and \(m_x\). Also, at the edge AB the yield moment is either zero if the edge is simply supported or \(m_x'\) if the edge is fixed against rotation. In the y direction strips membrane forces do exist and the actions at yield sections are \(M_y', M_y\) and \(N_y\). It can be seen that the actions at the yield sections in the x direction strips are independent of \(z\), but in the y direction strips they are dependent upon \(z\).

Suppose that the slab is given a downward virtual displacement \(\delta\) at the yield line EF. The slab segments, made up of portions of strips bounded by lines of yielding, will undergo virtual rotations about the lines of yielding at the edges of the slab.

Virtual rotation of segment ABFE about AB = \(\frac{\delta}{L_1}\)

Virtual rotation of segment DCFE about DC = \(\frac{\delta}{L_x-L_1}\)

Virtual rotation of segments ADE and BCF about AD and BC respectively = \(\frac{\delta}{L_2}\).

Let the strips in the x direction be dy in width and the strips in the y direction be dx in width.
Then the virtual work done by the internal actions at
the yield sections of the system of strips in the x direction

= Virtual work done by end portions of strips in
segments ABFE and DCFE only (since internal actions in
x direction strips in segments ADE and BCF have no corres­
ponding displacement)

\[
= - m^*_x \frac{\delta}{L_y} - (m'_x + m_x) \frac{\delta}{L_x - L_4} L_y. \quad (2.41)
\]

Where \( m^*_x \) depends upon the support conditions at the laterally
unrestrained edge:

If AB is simply supported \( m^*_x = m_x \) \( (2.42) \)

If AB is fixed against rotation \( m^*_x = m'_x + m_x \).

Also, the virtual work done by the internal :actions at
the yield sections of the system of strips in the y direction,

= Virtual work done by the end portions of strips in
segments ADE and BCF only (since internal actions in y
direction strips in segments ABFE and DCFE have no corres­
ponding displacement), and from (2.32)

\[
= -2 \int_0^{L_4} (\alpha_y + \beta_y z + yz^2) \frac{\delta}{L_2} \, dx -2 \int_{L_4}^{L_x} (\alpha_y + \beta_y z + yz^2) \frac{\delta}{L_2} \, dx
\]
\[
L - x - y(\frac{x}{L_x} + \gamma \frac{x^2}{L_x^2}) \frac{\Delta}{L_1} dx - 2 \int_{L_1}^{L_x} \left[ \alpha_y + \beta_y \left( \frac{L_x - x}{L_x - L_1} \right) \Delta \right.
\]
\[
+ \gamma \left( \frac{L_x - x}{L_x - L_1} \right)^2 \Delta^2 \left. \right] \frac{\Delta}{L_2} dx
\]
\[
= - \frac{2\Delta}{L_2} \left( L_x \alpha_y + \frac{L_x - x}{2} \beta_y + \frac{L_x}{3} \Delta \right).
\] (2.43)

Also the virtual work done by the external loading, \( w \) per unit area, from (2.30)
\[
= w[2L_x L_2 \frac{\Delta}{3} + (L_y - 2L_2)L_x \frac{\Delta}{2}]
\]
\[
= \frac{w\Delta L}{6} (3L_y - 2L_2).
\] (2.44)

Now from the equation of virtual work, (2.33),
\[
(2.44) + (2.41) + (2.43) = 0.
\]

From which \( \Delta \) cancels and the following expression for \( w \) is determined for \( L_y > L_x \):
\[
w = \frac{6}{L_x (3L_y - 2L_2)} \left[ L_y \left( \frac{m^*}{L_1} + \frac{m^* + m}{L_x - L_1} \right) + \frac{2L_x}{L_2} \alpha_y + \frac{L_x}{L_2} \beta_y + \frac{L_x}{3} \Delta \right].
\] (2.45)

The values of \( L_1 \) and \( L_2 \) required for (2.45) are those values which give minimum \( w \) at a given \( \Delta \), and may be found from:
\[
\frac{\partial w}{\partial L_1} = \frac{\partial w}{\partial L_2} = 0.
\]

The result of these operations is to give:

\[
\frac{L_1}{L_x} = \frac{\sqrt{m_x^*}}{\sqrt{m_x^*} + \sqrt{m_x^* + m_z}}.
\] (2.46)

And \(L_0\) is given as the solution of the following quadratic equation:

\[
0 = \left( \frac{L_0}{L_x} \right)^2 L_y \left( \frac{m_x^*}{L_1} + \frac{m_x^* + m_z}{L_x - L_1} \right) + \left( \frac{4L_0}{L_x} - \frac{3L_y}{L_x} \right) \left( \alpha_y + \frac{A}{2}, \beta_y + \frac{A^2}{3}y \right). \] (2.47)

Where in (2.45), (2.46) and (2.47), \(m_x^*\) is given by (2.42) for the appropriate support conditions at edge AB.

b) **Alternative yield-line pattern 2:**

The yield-line pattern is shown in Fig. 2.6b. The positions of the yield lines are defined by dimensions \(L_0\) and \(L_3\). Due to symmetry EF lies in the centre of the slab.

The origin of coordinates is at A. If \(z\) is the vertical deflection at any point with coordinates \((x, y)\) at the yield sections of the strips, and \(z = \Delta\) at the line EF, then:

\[
z = \frac{x\Delta}{L_0} = \frac{2y\Delta}{L_y}, \text{ at any point on line AF}
\]
The deflection at points on lines BF and CE follow from the above by symmetry.

The actions at the yield sections of the x and y direction strips are as for alternative yield-line pattern 1.

Suppose that the slab is given a downward virtual displacement \( \delta \) at the yield line EF. The slab segments, made up of portions of strips bounded by lines of yielding, will undergo virtual rotations about the lines of yielding at the edges of the slab.

- Virtual rotation of segment ABF about AB = \( \frac{\delta}{L_3} \)
- Virtual rotation of segment CDE about CD = \( \frac{\delta}{L_2} \)
- Virtual rotations of segments ADEF and BCEF about AB and BC respectively = \( \frac{2\delta}{L_y} \).

Let the strips in the x direction be dy in width and the strips in the y direction be dx in width.

Then the virtual work done by the internal actions at the yield sections of the system of strips in the x direction = Virtual work done by end portions of strips in segments ABF and CDE only (since internal actions in x direction strips in segments ADEF and BCEF have no
corresponding displacement)

\[ = -m^* \frac{\Delta}{L_3} L_y - (m'_x + m''_x) \frac{\Delta}{L_2} L_y. \] (2.48)

Where \( m^*_x \) depends upon the support conditions at edge AB and is given by (2.42).

Also, the virtual work done by the internal actions at the yield sections of the system of strips in the \( y \) direction

\[ = \text{Virtual work done by the end portions of the strips in segments ADEF and BCEF only (since internal actions in } \ y \text{ direction strips in segments ABF and CDE have no corresponding displacement), and from (2.32)} \]

\[ = -2 \int_0^{L_3} \left( \alpha_y + \beta_y z + \gamma z^2 \right) \frac{2 \delta}{L_y} dx - 2(\alpha_y + \beta_y z + \gamma z^2) \frac{2 \delta}{L_y} (L_x - L_a - L_3) \]

\[ -2 \int_{L_x - L_a}^{L_x} \left( \alpha_y + \beta_y z + \gamma z^2 \right) \frac{2 \delta}{L_y} dx \]

\[ = -2 \int_0^{L_3} \left( \alpha_y + \beta_y \frac{L_3}{L_3} + \gamma \frac{x^2 \Delta^2}{L_3^2} \right) \frac{2 \delta}{L_y} dx - 2(\alpha_y + \beta_y \Delta + \gamma \Delta^2) \frac{2 \delta}{L_y} (L_x - L_a - L_3) \]

\[ -2 \int_{L_x - L_a}^{L_x} \left\{ \alpha_y + \beta_y \frac{(L_x - x) \Delta}{L_3} + \gamma \frac{(L_x - x)^2 \Delta^2}{L_3^2} \right\} \frac{2 \delta}{L_y} dx \]

\[ = - \frac{L_3}{L_y} \left\{ L_x \alpha_y + \Delta(L_x - \frac{L_3}{2} - \frac{L_3}{2}) \beta_y + \Delta^2(L_x - \frac{2}{3} L_a - \frac{2}{3} L_3) \gamma \right\} \] (2.49)
Also the virtual work done by the external loading, \( w \) per unit area, from (2.30)

\[
= w\left\{ L_y (L_2 + L_3) \frac{\Delta}{3} + (L_x - L_2 - L_3) L_y \frac{\Delta}{2} \right\}
\]

\[
= \frac{w\delta L}{6} (3L_x - L_2 - L_3) \tag{2.50}
\]

Now from the equation of virtual work, (2.33),

\[
(2.50) + (2.48) + (2.49) = 0
\]

From which \( \delta \) cancels and the following expression for \( w \) is determined for when \( L_y > L_x \):

\[
w = \frac{6}{L_y (3L_x - L_2 - L_3)} \left\{ L_y \left( \frac{m^*}{L_3} + \frac{m' + m}{L_2} \right) + \frac{4L_x}{L_y} \alpha_y 
+ 2\Delta \frac{(2L_x - L_2 - L_3)}{L_y} \beta_y + \frac{4}{3} \Delta^2 \frac{(3L_x - 2L_2 - 2L_3)}{L_y} \gamma \right\} \tag{2.51}
\]

The values of \( L_2 \) and \( L_3 \) required for (2.51) are found from:

\[
\frac{\partial w}{\partial L_2} = \frac{\partial w}{\partial L_3} = 0.
\]

The result of these operations is to give:
\[ L_3 = L_a \sqrt{\frac{m^*_x}{m_x + m_x}}. \quad (2.52) \]

And \( L_a \) is given by the solution of the following quadratic equation:

\[
0 = \left( \frac{L_a}{L_y} \right)^2 \frac{2L_x}{L_y} (2\alpha_y - \Delta_y - 2\Delta^2 y) + \frac{2L_a}{L_y} (\sqrt{m^*_x \left( m_x + m_x \right)} + m_x' + m_x') \\
- \frac{3L_x}{L_y} (m_x' + m_x). \quad (2.53)
\]

Where, in (2.51), (2.52) and (2.53), \( m^*_x \) is given by (2.42) for the appropriate support conditions at edge AB.

(iii) A short edge unrestrained laterally: Let the slab be loaded beyond the point of ultimate load so that the maximum deflection of the slab lies somewhere between the points B and C of Fig.1.1. The alternative yield-line patterns are shown in Fig.2.7 for the case of the unrestrained edge being simply supported. Had the unrestrained edge been fixed against rotation the only addition to the alternative yield-line patterns would be a hogging moment yield line along that edge. Comparison of Fig.2.6 and Fig.2.7 shows that the expressions for short edge unrestrained laterally may be obtained directly from the results already obtained for a long edge unrestrained laterally by interchanging the
x and y subscripts for dimensions and by interchanging the coefficients and yield moments which define the actions at yield sections of the x and y direction strips:

a) **Alternative yield-line pattern 1:**

For \( L_Y > L_X \), from (2.45):

\[
\begin{align*}
w &= \frac{6}{L_Y(3L_X-2L_Z)} \left[ L_X \left( \frac{m^*_Y}{L_1} + \frac{m^+_Y + m^-_Y}{L_Y - L_1} \right) + \frac{2L_Y}{L_2} \alpha_X + \Delta \frac{L_Y}{L_2} \beta_X \right. \\
&\left. + \Delta^2 \frac{2}{3} \frac{L_Y}{L_2} \right].
\end{align*}
\]  

(2.54)

Where for minimum \( w \) at a given \( \Delta \) the values of \( L_1 \) and \( L_2 \) are given by:

From (2.46):

\[
\frac{L_1}{L_Y} = \frac{\sqrt{m^*_Y}}{\sqrt{m^*_Y} + \sqrt{m^+_Y + m^-_Y}}
\]  

(2.55)

From (2.47):

\[
o = \left( \frac{L_2}{L_Y} \right)^2 L_X \left( \frac{m^*_Y}{L_1} + \frac{m^+_Y + m^-_Y}{L_Y - L_1} \right) + \left( \frac{L_2}{L_Y} - \frac{L_X}{3L_Y} \right) (\alpha_X + \Delta \frac{\beta_X}{2} + \Delta^2 \frac{\gamma}{3})
\]  

(2.56)

Where \( m^*_Y \) depends upon the support conditions at the laterally unrestrained edge:
If AB is simply supported, \( m_y^* = m_y \)  \( \quad (2.57) \)

If AB is fixed against rotation, \( m_y^* = m_y' + m_y \)

b) **Alternative yield-line pattern 2:**

For \( L_y > L_x \), from (2.51):

\[
w = \frac{6}{L_x(3L_y-L_2-L_3)} \left( L_x \left( \frac{m^*_y}{L_3} \right) + \frac{m_y'}{L_2} \right) + \frac{L_y}{L_x} \alpha_x \]

\[
+ 2\Delta \left( \frac{2L_y-L_2-L_3}{L_x} \right) \beta_x + \frac{1}{3} \Delta^2 \left( \frac{3L_y-2L_2-2L_3}{L_x} \right) \gamma \]

\( \quad (2.58) \)

Where for minimum \( w \) at a given \( \Delta \) the values of \( L_2 \) and \( L_3 \) are given by:

From (2.52):

\[
L_3 = L_2 \sqrt{\frac{m^*_y}{m_y' + m_y}} \]

\( \quad (2.59) \)

From (2.53):

\[
0 = \left( \frac{L_2}{L_x} \right)^a \left( \frac{2L_y}{L_x} \left( 2\alpha_x - \Delta \beta_x - 2\Delta^2 \gamma \right) + \frac{L_2}{L_x} \left( \sqrt{m^*_y (m_y' + m_y)} + m_y' + m_y \right) \right)

- \frac{3L_y}{L_x} (m_y' + m_y) \]

\( \quad (2.60) \)

Where in (2.58), (2.59) and (2.60), \( m_y^* \) is given by (2.57) for the appropriate support conditions of edge AB.
(iv) The critical yield-line pattern and an approximation to the load carried by the slab: To analyse a given slab using the equations derived, the dimensions defining the configuration of yield lines giving minimum \( w \) at a given \( \Delta \) are first found for each alternative yield-line pattern. These dimensions are then substituted into the equations for \( w \) and the load \( w \) sought is the minimum value given at a particular \( \Delta \) by the two alternative yield-line patterns. Usually it will be found that the configuration of yield lines required for one alternative is impossible, for example (2.47) may require \( L_2 > 0.5L_y \) in Fig.2.6a, and the minimum load is then given by the other alternative collapse mechanism.

It is evident that the determination of \( w \) using the above procedure leads to very lengthy calculations and a means of simplifying the procedure will now be examined. It is shown in Appendix A that in the case of Johansen's yield-line theory if it is assumed that the sagging moment yield lines which run into the corners of the slab are at \( 45^\circ \) to the edges of the slab the error in the ultimate load \( w_J \) found using the virtual work method is generally less than 7\% for uniformly loaded slabs which contain practical amounts of reinforcement and which have one edge simply
supported and the other edges fixed against rotation. This approximate yield-line pattern leads to vast simplification in the calculations involved. The accuracy of the assumption of $45^\circ$ sagging moment yield lines in the corners will now be examined for the case including membrane stresses. Typical lightly reinforced slabs (containing the minimum amount of reinforcement allowed by C.P.114: 1957\textsuperscript{1}, 0.15\% of the concrete cross-section, since then the effect of membrane stresses will be greatest) with various $\frac{L_y}{L_x}$ ratios, cube strength of 3,000 lb./sq.in., steel yield stress of 40,000 lb./sq.in., effective depth of steel of 0.8 of the slab thickness, no compression steel and the laterally unrestrained edge simply supported, will be considered. For such slabs the yield-line patterns and the loads at central deflections of 0.4 and 0.6 of the slab thickness (the deflections which are expected to be near the ultimate flexural load in the curve B to C of Fig.1.1) are compared in Table 2.1. The "exact" w values have been found from the equations derived, and the "approximate" w values have been found by substituting $L_y = L_o = 0.5L_x$ into (2.45) for the case of a long edge simply supported and $L_o = L_o = 0.5L_x$ into (2.58) for the case of a short edge simply supported. The table shows that
<table>
<thead>
<tr>
<th>$\frac{L_y}{L_x}$</th>
<th>$A_d$</th>
<th>A Long Edge Simply Supported</th>
<th>A Short Edge Simply Supported</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Alternative Pattern 1</td>
<td>Alternative Pattern 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fig. 2.6a.</td>
<td>Fig. 2.6b.</td>
</tr>
<tr>
<td>$L_1/L_x$</td>
<td>$L_2/L_y$</td>
<td>$L_3/L_y$</td>
<td>$L_1/L_y$</td>
</tr>
<tr>
<td>(2.46)</td>
<td>(2.53)</td>
<td>(2.52) &amp; (2.53)</td>
<td>(2.55)</td>
</tr>
<tr>
<td>1.0</td>
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<td>0.265</td>
<td>0.265</td>
</tr>
<tr>
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<td>0.262</td>
<td>0.243</td>
</tr>
<tr>
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<td>0.240</td>
</tr>
<tr>
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<td>0.223</td>
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<td>0.211</td>
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<td>0.144</td>
</tr>
<tr>
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<td>0.144</td>
<td>0.159</td>
</tr>
<tr>
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<td>0.144*</td>
<td>0.144*</td>
</tr>
<tr>
<td>3.0</td>
<td>0.6</td>
<td>0.144*</td>
<td>0.154*</td>
</tr>
</tbody>
</table>

* Governing yield-line pattern in cases when both alternatives are possible.
theoretically the yield-line pattern varies with the $\Delta d$ ratio for a given slab, but the variation is small. Only one concrete strength has been considered but it can be shown that an increase in concrete strength makes little difference to the ratio of $\frac{\text{Approximate} \ w}{\text{Exact} \ w}$ calculated. The error shown by the ratio $\frac{\text{Approximate} \ w}{\text{Exact} \ w}$ is large in most cases and the approximate expressions over-estimate the load carried. This over-estimation is to be expected since an upper bound approach has been used in the theory. However, in view of the large saving of calculation resulting from the approximation it may be thought that the approximate equations are sufficiently accurate for design purposes, especially since a correction factor of the order of the inaccuracy shown in the table could be used. Other factors which favour the use of the approximate equations are firstly, that the assumption of zero membrane forces acting in the direction normal to the simply supported edge is conservative (some are bound to develop), and hence the "exact" expressions are bound to underestimate the load which would be found experimentally. Secondly, for more heavily reinforced slabs than checked in the above table the membrane forces become smaller and the discrepancy between the "exact" and the "approximate" loads
is reduced. For very heavily reinforced slabs the
discrepancy would tend towards the less than 7% given
when the approximate yield-line pattern is used for
Johansen's yield-line theory.

The approximate forms of the equations for w
(which have already been used in the above table) will now
be written:

a) **A long edge unrestrained laterally, remaining edges
fully fixed:**

The assumed yield-line pattern is as in Fig.2.8b.
The approximate \( w \) is found from (2.45) with \( L_t = L_x = 0.5L_x \).

\[
w = \frac{2h}{L_x^2 \left( \frac{L_y}{L_x} - 1 \right)} \left\{ \frac{1}{2} \frac{L_y}{L_x} (m_x^* + m_y^* + m_x) + \alpha_y + \frac{1}{2} \Delta \beta_y + \frac{1}{3} \Delta^2 y \right\} \tag{2.61}
\]

Where \( m_x^* = m_x \) if the laterally unrestrained edge is simply
supported, or \( m_x^* = m_x^* + m_x \) if the edge is fixed against
rotation.

b) **A short edge unrestrained laterally, remaining edges
fully fixed:**

The assumed yield-line pattern is as shown in Fig.2.8c.
The approximate \( w \) is found from (2.58) with \( L_a = L_z = 0.5L_x \).

\[
w = \frac{2h}{L_x^2 \left( \frac{L_y}{L_x} - 1 \right)} \left\{ \frac{1}{2} (m_y^* + m_y^* + m_y) + \frac{L_y}{L_x} \alpha_y + \Delta \left( \frac{L_y}{L_x} - \frac{1}{2} \right) \beta_y + \frac{1}{3} \Delta^2 \left( \frac{3L_y}{L_x} - 2 \right) y \right\} \tag{2.62}
\]
FIG. 2.8 APPROXIMATE YIELD LINE PATTERNS FOR UNIFORMLY LOADED TWO-WAY SLABS.

FIG. 2.9 UNIFORMLY LOADED FIXED-ENDED REINFORCED CONCRETE BEAM AT ULTIMATE LOAD.
Where \( m^* = m_y \) if the laterally unrestrained edge is simply supported, or \( m^* = m'_y + m_y \) if the edge is fixed against rotation.

2.3 Theory for the Deflection of the Slab at the Ultimate Flexural Load.

2.3.1 Conditions at Yield Lines at the Ultimate Flexural Load

To obtain the theoretical ultimate flexural load of the slab from the equations already developed relating \( w \) and \( \Delta \), the value of \( \Delta \) at the ultimate flexural load, \( \Delta_u' \), must be known. This means determining the deflection at the point B on the curve of Fig.1.1, the deflection at which the yield-line pattern of the slab has just developed. The development of the yield-line pattern requires the deformation of the slab to be great enough to cause sufficiently large strains at the yield lines to force the tension steel to yield and the compressed concrete to reach its ultimate value. All yield lines will not reach this condition of plasticity at the same load and the slab will reach the ultimate flexural load when the deflection is just sufficient to cause the last yield line of the collapse mechanism to form.
Before commencing to develop a theory to determine $\Delta u$, it should be noted that when a slab is loaded to failure it forms a collapse mechanism which is made up from plane segments connected by bands of yielding of finite width rather than by lines of yielding. Hence a yield line is really an idealization for a yield band. Assuming that all rotation occurs along lines makes little difference when relating collapse loads and yield moments, but to determine the deflection of the slab at ultimate flexural load yield bands will have to be considered. To look more closely at the formation of plastic hinges in reinforced concrete flexural members, Fig. 2.9 shows the collapse mechanism of a uniformly loaded fixed-ended beam in which both steel and concrete have become plastic at the regions of maximum positive and negative bending moment. It can be seen that the lengths of the regions of the beam which contribute to the rotation of each "hinge" are quite extensive. The length of the region of yielding at the centre of the beam is greater than that at the ends of the beam due to the shape of the bending moment diagram. The sagging moment is near maximum over a good portion of the centre of the beam whereas the hogging bending moment rises to a sharp peak at each end.
A semi-empirical method for determining the deflection at failure of slabs without membrane action has been reported by Korolev\textsuperscript{11} and good correlation with a limited number of test results was claimed. An approach similar to that used by Korolev will now be used to determine the deflection of slabs with membrane action at the ultimate flexural load. This will involve finding the value of $\Delta$ at which the last yield-line of the collapse mechanism has just developed.

2.3.2 Uniformly Loaded Slab With All Edges Fixed Against Rotation and Translation.

In the theory the following assumptions will be made:

1. The sagging moment yield lines entering into the corners of the slab are at $45^\circ$ to the edges of the slab. (This approximate yield-line pattern was adopted when determining the load-deflection curves in previous sections).

2. The four segments of the slab making up the portions of the collapse mechanism are plane, and hence elastic deformations are neglected.

3. The regions of plasticity between the segments occur in bands and the curvature of the slab within the bands is constant at the maximum value occurring in the slab.
Hence the actual variation in curvature has been replaced by a rectangle (see Fig.2.9) with the height of the rectangle equal to the maximum curvature. The width of the band of yielding has a constant value $L_p$ for the sagging moment yield bands and a constant value $L'_p$ for the hogging moment yield bands, as in Fig.2.10. $L'_p$ will be less than $L_p$ due to the variation in bending moment as in the case of the beam of Fig.2.9.

Consider the stage at which the slab has just reached its ultimate flexural load. At this stage the collapse mechanism of the slab will have just formed and at all yield bands the tension steel will be at yield stress and the strain at the compression face of the concrete, $e_c$, will be approximately of the magnitude shown in Fig.2.2 for the appropriate cube strength. Referring to Fig.2.10 it can be seen that:

Rotation of slab segments about hogging moment yield bands

$$AB, \ BC, \ CD \ and \ DA = \frac{2\Delta u}{L_x}$$  \hspace{1cm} (2.63)

Total rotation of slab segments about sagging moment yield band $EF = \frac{4\Delta u}{L_x}$  \hspace{1cm} (2.64)

Total rotation of slab segments about sagging moment yield bands $AE, BF, CF$ and $DE = \frac{2\sqrt{2}\Delta u}{L_x}$  \hspace{1cm} (2.65)
FIG. 2.10 YIELD BANDS OF A UNIFORMLY LOADED TWO-WAY SLAB WITH ALL EDGES FIXED.

FIG. 2.11 STRAIN DISTRIBUTION AT YIELD SECTIONS.
It can be seen that the last sagging moment yield bands to develop will be AE, BF, CF and DE (since the rotations there are less than at band EF), and hence band EF need no longer be considered. It is unknown, however, whether the last band of the mechanism to form is a sagging moment or hogging moment band as they are of different widths. Now consider the strain diagrams for yield sections shown in Fig.2.11. Since when the slab is at ultimate flexural load the neutral axis depths will almost always be less than one half of the slab thickness (as shown by (2.8) and (2.9)) and the strain in the concrete in the compressed faces will usually be greater than 0.003, it is evident that all the tension steel will have yielded before the concrete reaches its required deformation. Hence the concrete strain will govern the stage at which it may be considered that the last yield band has developed. The rotations when the concrete has just reached the required strain to develop maximum moment of resistance is, from Fig.2.11:

Rotation of slab segments about hogging moment yield bands

\[ \frac{L'e_{C}}{n_{d}} \]  

(2.66)
Rotation of slab segments about sagging moment yield bands

\[ \frac{L_e c}{n_i d} \]  \hspace{1cm} (2.67)

Hence if the hogging moment yield bands form last, (2.63) = (2.66).

\[ \Delta_u = \frac{e L' L}{2n_i d} \]  \hspace{1cm} (2.68)

And if the sagging moment yield bands form last, (2.65) = (2.67).

\[ \Delta_u = \frac{e L L}{2\sqrt{2} n_i d} \]  \hspace{1cm} (2.69)

Examination of the equations for neutral axis depth found by the strip approximation, (2.8) and (2.9), shows that the steel in typical lightly reinforced slabs makes little difference to the neutral axis positions, and the equations may be rewritten approximately as:

\[ n_i' = n_i = 0.5(1 - \frac{z}{2d}) \]  \hspace{1cm} (2.70)

(2.70) shows that the neutral axis depth varies over the slab. The yield bands will develop last where the neutral
axis depth is a minimum (since the smaller the neutral axis depth the greater the rotation must be to produce a given strain at the compressed edge), and hence the values of \( n \) and \( n_1 \) required for (2.68) and (2.69) are given by substituting \( z = \Lambda_u \) into (2.70). (2.68) and (2.69) then become:

If hogging moment yield bands form last, from (2.68):

\[
\Lambda_u = \frac{e \frac{L_1 L}{c p x}}{(1 - \frac{\Lambda_u}{2d})d}, \quad \text{or} \quad \Lambda_u = d\left(1 - \sqrt{1 - 2e \frac{L_1 L}{c d d}}\right) \quad (2.71)
\]

If sagging moment yield bands form last, from (2.69):

\[
\Lambda_u = \frac{e \frac{L_1 L}{c p x}}{\sqrt{2}(1 - \frac{\Lambda_u}{2d})d}, \quad \text{or} \quad \Lambda_u = d\left(1 - \sqrt{2e \frac{L_1 L}{c d d}}\right) \quad (2.72)
\]

Now if the sagging moment yield bands form after the hogging moment yield bands (2.72)>(2.71), which requires \( 0.7L_p > L_1' \).

Examination of the distribution of bending moment across the slab (which is similar to that shown for the beam of Fig.2.9) shows that it is likely that \( L_p > L_1' \), due to the rapid decrease of hogging moment from the maximum at the slab edges and the slow decrease of sagging bending moment from the maximum at the centre. Hence the sagging moment yield bands AE, BF,
CF and DE will be the last to develop and the deflection of the slab at the ultimate flexural load is given by (2.72), which may be rewritten as:

\[
\frac{\Delta u}{d} = 1 - \sqrt{1 - 1.414e c \frac{L_p}{d} \frac{L}{d}}
\]  

(2.73)

In (2.73), \( e_c \) is given by Fig. 2.2 and \( \frac{L}{d} \) is known for a particular slab. \( \frac{L_p}{d} \) depends upon the distribution of bending moment but there does not appear to be a satisfactory method for determining it theoretically. Experimental results will be examined later in order to determine whether an empirical value for \( \frac{L_p}{d} \) can be established.

### 2.3.3 Uniformly Loaded Slab With Three Edges Fixed Against Rotation and Translation and the Remaining Edge Free to Translate Horizontally.

This case differs from that of all edges fully fixed in that the membrane forces acting in the direction normal to the laterally unrestrained edge will be small, if not zero. In the case of all edges fully fixed, the neutral axis depth in both directions was given approximately by (2.70). In this case, however, for steel running normal to the laterally unrestrained edge the neutral axis will be close to the compressed face of the slab, and only in
the other direction will the position of the neutral axis be given approximately by (2.70). Also, with one edge simply supported the yield-line pattern varies sufficiently from the approximate pattern with 45° corner lines to make deflection calculations from the approximate pattern too inaccurate to be considered. No attempt will be made to determine an expression for \( \frac{\Delta u}{d} \) for this case since taking into account the variable yield-line pattern and neutral axis depths would lead to an unwieldy expression which would be too inaccurate to justify the large amount of computation involved in its use. Test results will be examined later to determine an empirical value for \( \frac{\Delta u}{d} \) for this case.

2.4 Experimental Work

2.4.1 Range of Experimental Results Available from Past Investigations.

Powell\(^5\) and Wood\(^6\) have reported results of tests conducted on slabs with all edges fully restrained. It was evident that more test results were required for the case of all edges fully restrained, especially for slabs with different \( L_y/L_x \) ratios and various \( L_x/d \) ratios, and also for the case when one edge is free to translate horizontally.
To extend the range of available test results four series of slabs were tested.

2.4.2 The Testing Arrangements and the Slabs

The details of the testing frame and method of loading used are described in Appendix B. It is to be noted that for convenience the slabs were tested under upward pressure applied to the bottom face. In the theoretical sections the usual case of downward loading has been considered when referring to sagging and hogging bending moments. To avoid confusion the details of the test slabs will be referred to as if the test set-up had not been inverted. For example reinforcement that was actually in the bottom of the slab when tested will be referred to as top steel, and edge moments will be referred to as hogging moments.

The details of the slabs are given in full in Appendix B and are summarized in Tables 2.3 and 2.4. The reinforcement was of annealed mild steel to ensure that the yield stress was well marked. The area of each slab was larger than the tested area to provide a 12\(\frac{1}{2}\)in. edge strip for clamping purposes for fully restrained edges, or a lin. overhang for simply supported edges. The slabs of series A, B and C were reinforced as if designed by Johansen's
yield-line theory for minimum weight of reinforcement at various ultimate loads. The top steel only extended into the slab from the edges by distances which were calculated by Johansen's yield-line theory to be sufficient to ensure that hogging moment yield lines formed along the lines of the supports. The bottom steel extended over the whole of each slab. Appendix A shows the method of designing two-way slabs by Johansen yield-line theory for minimum weight of steel and for curtailed top steel. The slabs of series D were unreinforced. The slabs of series A and D were tested with all edges fully restrained against movement, and the slabs of series B and C were tested with three edges fully restrained against movement and one edge, either a short or a long edge, simply supported.

The load was applied to the slabs in increments in order to measure strains and deflections and the time taken to load each slab up to its ultimate load varied between 1½ and 2½ hours. Loading of the reinforced slabs was continued as far as possible into the tensile membrane stage.

Figure 2.12 shows the general test set-up. Figures 2.13, 2.14, 2.15 and 2.16 show the yield-line patterns developed in the slabs at the end of the test runs. Since the plates were taken after tensile membrane action had also
FIG. 2.12 A SLAB IN THE FRAME BEFORE TESTING.
FIG. 2.13 LOADED FACES OF SLAB SERIES A AT END OF TEST.
FIG. 2.14 LOADED FACES OF SLAB SERIES B AT END OF TEST.
FIG. 2.15 LOADED FACES OF SLAB SERIES C AT END OF TEST.
Cracks within the segments of the yield-line patterns were formed after the testing. For slabs D3, D4 and D5 the segments are shown separated for clarity.

FIG. 2.16 LOADED FACES OF SLAB SERIES D AT END OF TEST.
occurred the figures also show cracks which were formed after the ultimate flexural load had been reached, but the lines of crushing indicate the yield-line pattern.

2.5 **Comparison of Theoretical and Experimental Load-Deflection Curves.**

2.5.1 **Slabs With All Edges Fully Restrained**

The slabs of series A (reinforced) and D (unreinforced) gave the curves of uniformly applied load versus central deflection shown in Figs. 2.17 and 2.18. Also plotted in the figures are theoretical curves given by (2.40) and (4.9). Reference will be made to the theoretical curves given by (4.9) in Chapter 4.

The theoretical curves given by (2.40), defining the descending portion after ultimate flexural load, can only be expected to apply near the point of ultimate flexural load since beyond this point the concrete at yield lines has crushed and is beginning to disintegrate and so cause the area of the section to be reduced. It can be seen that the curves are conservative in most cases. In order to use (2.40) to estimate the ultimate flexural load the central deflection at this load is required. (2.73) was derived for this purpose, but the equation depends upon a value
FIG. 2.17 LOAD-DEFLECTION CURVES FROM SLAB SERIES A.
REINFORCED SLABS WITH ALL EDGES FULLY FIXED.
FIG. 2.18 LOAD-DEFLECTION CURVES FROM SLAB SERIES D, UNREINFORCED SLABS WITH ALL EDGES FULLY FIXED.
for $\frac{L_p}{d}$ which is to be found empirically. Table 2.2 shows the test results of Powell, Wood and the slabs of series A and D analysed to determine $\frac{L_p}{d}$ from (2.73). The steel contents of the slabs are given in Table 2.3 and it is evident that the $\frac{L_p}{d}$ ratio is independent of the amount of reinforcement in the slab, as was indicated by the theory.

Table 2.2 shows that there is a relationship between $\frac{L_p}{d}$ and $\frac{L_X}{d}, \frac{L_p}{d}$ decreasing as $\frac{L_X}{d}$ increases. This trend is illustrated in Fig. 2.19. In order to obtain a conservative value for the ultimate flexural load from (2.40) the maximum value of $\frac{A_u}{d}$ likely to occur is required. Hence the value of $\frac{L_p}{d}$ required for (2.73) is also the maximum value likely to occur for a particular slab. An empirical second order equation which defines the upper limit of scatter of the points plotted in Fig. 2.19 is:

$$\frac{L_p}{d} = 18.4 - 0.65 \frac{L_X}{d} + \frac{1}{138} \left( \frac{L_X}{d} \right)^2, \text{ for } 16 < \frac{L_X}{d} < 40 \text{ (2.74)}$$

On substituting (2.74) into (2.73) the following equation for the maximum deflection at the ultimate flexural load results:

$$\frac{A_u}{d} = 1 - \sqrt{1 - 1.41e_c \frac{L_X}{d} \left[ 18.4 - 0.65 \frac{L_X}{d} + \frac{1}{138} \left( \frac{L_X}{d} \right)^2 \right]} \text{ (2.75)}$$
<table>
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<th>Authority</th>
<th>Slab Mark</th>
<th>(\frac{L}{d})</th>
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<th>(\frac{\Delta u}{d}) from tests</th>
<th>(\frac{L}{d}) from (2.73)</th>
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Note: Details of dimensions, cube strengths and reinforcement contents for the slabs are given in Table 2.3.
FIG. 2.19 VARIATION IN WIDTH OF YIELD BANDS OF UNIFORMLY LOADED TWO-WAY SLABS WITH FULLY RESTRAINED EDGES.
On substituting (2.75) into (2.40) a conservative value for the ultimate flexural load should result if the theoretical and experimental load-deflection curves coincide. It is evident, however, that the $\frac{L_p}{d}$ values calculated in Table 2.2 are surprisingly large. In the slabs of series A ($L_y = 60$in., $L_x = 40$in.) the yield bands are calculated to be 16 to 20in. wide. Yielding of this extent was not observed in these slabs at the ultimate flexural load and it is suspected that the values of $e_c$ used in the calculations (from the test results of Hognestad, Hanson and McHendry\textsuperscript{10} on specimens under uniaxial stress) were low for the biaxial stress conditions present in the slabs. If higher values of $e_c$ were to be used in the calculations smaller values of $\frac{L_p}{d}$ would result. However, since it is the product of $e_c$ and $\frac{L_p}{d}$ that is required in (2.73), and not the individual values, no attempt will be made to modify these quantities.

An alternative method for determining an empirical value for $\frac{Au}{d}$ is by direct inspection. Table 2.2 shows that the three groups of slabs reached ultimate flexural load with $\frac{Au}{d}$ in the range 0.33 to 0.50. This is a surprisingly narrow range since the slabs covered the wide range of $\frac{L_x}{d}$ and $\frac{L_y}{L_x}$ ratios shown in Table 2.3. Thus it would seem that a good estimate of the ultimate flexural load would result if $\Delta u = 0.5d$ (taking the larger deflection to be conservative) were to be substituted into (2.40). This substitution is far more convenient than using (2.75), since (2.75) is rather unwieldy for practical use.
Taking $\Delta u = 0.5d$ may appear to be a crude approximation since it seems to exclude the effect of $e_c$ and the $\frac{L^x}{d}$ ratio. The explanation is as follows. Fig. 2.2 shows that $e_c$ is almost constant over the practical range of cube strengths and Fig. 2.19 shows that the product $\frac{\frac{L^x}{d}}{d}$ is also almost constant since the curve drawn through the upper limit of scatter is approximately hyperbolic. Hence $e_c \cdot \frac{\frac{L^x}{d}}{d}$ will be approximately constant and inspection of (2.73) (the equation from which (2.75) was derived) shows that therefore $\frac{\Delta u}{d}$ will have an approximately constant value for all slabs in the range considered above. That this constant value of $\frac{\Delta u}{d}$ is approximately 0.5 may be verified by substituting typical values for $e_c$ and $\frac{L^x}{d}$ into (2.75).

2.5.2 Slabs With One Edge Simply Supported, the Other Edges Fully Restrained.

The slabs of series B and C gave the curves of uniformly applied load versus central deflection shown in Figs. 2.20 and 2.21. Also plotted on the figures are the theoretical curves. Reference to the theoretical curves given by (4.10) and (4.11) will be made in Chapter 4.

The theoretical curves plotted for the descending portion of the load-deflection curves were obtained from the "approximate" equations in which the yield-line pattern with corner yield lines at $45^\circ$ to the slab edges was assumed, and also from the "exact" equations which considered the configuration of yield lines which gave the minimum load to
FIG. 2.20 LOAD-DEFLECTION CURVES FROM SLAB SERIES B. REINFORCED SLABS WITH A SHORT EDGE SIMPLY SUPPORTED, OTHER EDGES FULLY FIXED.
FIG. 2.21 LOAD-DEFLECTION CURVES FROM SLAB SERIES C.
REINFORCED SLABS WITH A LONG EDGE SIMPLY SUPPORTED, OTHER EDGES FULLY FIXED.
cause failure. The figures show that even the "approximate" equations are almost always conservative. This no doubt results from the neglect of membrane stresses in the direction at right angles to the simply supported edges. Some membrane forces were bound to develop in this direction due to the fact that the fully restrained edges adjacent to the simply supported edge were unable to increase in length. Since the "approximate" equations are conservative it is felt that there is no need to use the "exact" equations which are even more conservative, especially since extremely lengthy calculation is involved in using the "exact" equations. The experimental curves show that the central deflection at the ultimate flexural load was approximately 0.33d for a short edge simply supported and approximately 0.4d (except for slab Cl) for a long edge simply supported. Thus it would seem that substituting \( \Delta_u = 0.4d \) (taking the larger deflection to be conservative) into the theoretical equations would give a good estimate of the ultimate flexural load.

2.6 **Formulation of Equations for Ultimate Flexural Load.**

2.6.1 **Slabs With All Edges Fully Restrained**

If the values of \( \alpha_x \), \( \alpha_y \), \( \beta_x \), \( \beta_y \) and \( \gamma \) from (2.20), (2.24), (2.21), (2.25) and (2.22), respectively, are substituted into (2.40), the following equation for ultimate flexural load, \( w_u \) (at the point B of Fig.1.1), results for slabs with \( L_y > L_x \):

\[ \]
\[
\frac{w_u L_x^2 (3 \frac{L_y}{L_x} - 1)}{24} = k_1 k_3 u d^2 \left\{ \frac{1}{2} (1 - k_2) \left( 1 + \frac{L_y}{L_x} \right) \phi \left( \frac{\Delta u}{d} \right) \frac{1}{4} (2k_2 - 3) \frac{L_y}{L_x} \right. \\
\left. + \left( \frac{\Delta u}{d} \right) \frac{1}{8} (2 - k_2) \left( \frac{L_y}{L_x} - \frac{1}{2} \right) \right\}
\]

\[- \frac{k_2}{2k_1 k_3 u \frac{L_y}{L_x}} \left\{ \left( \frac{L_y}{L_x} - c' + c_s x \right)^2 + \left( \frac{L_y}{L_x} - c' + c_s y \right)^2 \right\}
\]

\[+ \frac{L_y}{L_x} \left\{ c_{sx} \left( \frac{d}{2} - d_{2x} \right) + c'_{sx} \left( \frac{d}{2} - d_{2x}' \right) + T_x (d_{1x} - \frac{d}{2}) \right. \\
\left. + T_x (d_{1x}' - \frac{d}{2}) \right\}
\]

\[+ c_{sy} \left( \frac{d}{2} - d_{2y} \right) + c'_{sy} \left( \frac{d}{2} - d_{2y}' \right) + T_y (d_{1y} - \frac{d}{2}) + T_y (d_{1y}' - \frac{d}{2}) \]

\[+ \left( \frac{\Delta u}{d} \right) \phi \left\{ (2 \frac{L_y}{L_x} - 1) \left( T_x' - c' + c_{sx} \right) \right. \\
\left. + (T_y' - c' + c_{sy}) \right\} \tag{2.76}
\]

Where: \( \frac{\Delta u}{d} = 1 - \sqrt{1 - 1.414 e_c \frac{L_y}{d} \left[ 18.4 - 0.65 \frac{L_x}{d} + \frac{1}{138} \left( \frac{L_x}{d} \right)^2 \right]} \) from (2.75)

Alternatively, if \( \frac{\Delta u}{d} = 0.5 \) is substituted into (2.76), as suggested at the end of section 2.5.1, the following equation for \( w_u \) results for slabs with \( L_y > L_x \):
\[
\frac{w_u L_x^2 (3 \frac{L_y}{L_x} - 1)}{24} = k_1 k_3 u d^2 \left\{ \frac{L_y}{L_x} (0.188 - 0.281 k_2) + (0.479 - 0.490 k_2) \right\} \\
- \frac{k_2}{2 k_1 k_3 u} \left\{ \frac{L_y}{L_x} \left( T_x' - T_x - C'_{sx} + C_{sx} \right)^2 + (T_y' - T_y - C'_{sy} + C_{sy})^2 \right\} \\
+ C_{sx} \left\{ \frac{L_y}{L_x} (\frac{d}{4} - d_2) + \frac{d}{8} \right\} + C'_{sx} \left\{ \frac{L_y}{L_x} (\frac{d}{4} - d_2') + \frac{d}{8} \right\} \\
+ T_x \left\{ \frac{L_y}{L_x} (d_1' - \frac{d}{4}) - \frac{d}{8} \right\} + T_x' \left\{ \frac{L_y}{L_x} (d_1' - \frac{d}{4}) - \frac{d}{8} \right\} \\
+ C_{sy} \left( \frac{3}{8} d - \frac{3}{8} d_2 y \right) + C'_{sy} \left( \frac{3}{8} d - \frac{3}{8} d_2' y \right) + T_y (d_1 y - \frac{3}{8} d) \\
+ T_y' (d_1 y' - \frac{3}{8} d') \right. 
\]

(2.77)

(2.76) and (2.77) will simplify considerably in most practical cases. For instance, in almost all slabs \(d_{1x}' = d_{1x}, \ d_{2x}' = d_{2x}, \ d_{1y} = d_{1y} \) and \(d_{2y}' = d_{2y}, \) and also if the slab has equal top and bottom steel placed over the whole slab, but not necessarily the same steel in both directions, then \(T_x = T_x' = C_{sx} = C_{sx}' \) and \(T_y = T_y' = C_{sy} = C_{sy}' \), and (2.77) then becomes:

\[
\frac{w_u L_x^2 (3 \frac{L_y}{L_x} - 1)}{24} = k_1 k_3 u d^2 \left\{ \frac{L_y}{L_x} (0.188 - 0.281 k_2) + (0.479 - 0.490 k_2) \right\} \\
+ 2 \left\{ \frac{L_y}{L_x} T_x (d_1 x - d_2 x) + T_y (d_1 y - d_2 y) \right\} 
\]

(2.78)
It is of interest to note that the left hand side of (2.78) equal to the second term of the right hand side of (2.78) is almost the Johansen yield-line theory expression for ultimate strength, and thus the first term on the right hand side represents the contribution of membrane action to the strength in this case.

Table 2.3 shows the comparison of the theoretical ultimate flexural loads determined from (2.75) and (2.76), and from (2.77), with the experimental ultimate flexural loads obtained from the slabs of Powell\(^5\), Wood\(^6\) and the series A and D. The equations are shown to predict with reasonable accuracy the ultimate flexural loads of these slabs, and it is to be noted that the slabs cover a range of \(\frac{L_Y}{L_X}\) ratios of 1.0, 1.5 and 1.75, \(\frac{L_X}{d}\) ratios of 16, 20, 26.7, 30.2 and approximately 40, and steel contents of zero to well over 1\%. Most practical two-way slabs will fall into these ranges. Table 2.3 also includes the theoretical ultimate flexural loads calculated by Johansen's yield-line theory and illustrates how it underestimates the ultimate strength of slabs with fully restrained boundary conditions. Table 2.3 also shows that the use of \(\frac{\Delta u}{d} = 0.5\) as the central deflection at ultimate flexural load, rather than the value given by (2.75), results in an estimate for the ultimate
<table>
<thead>
<tr>
<th>Authority</th>
<th>Slab Mark</th>
<th>Dimensions L x W x H in.</th>
<th>LW/Ld</th>
<th>% of Steel Reinforcement</th>
<th>Cube Strength Top</th>
<th>Extrap. U/L. w.e.</th>
<th>Ultimate U/L. w.e.</th>
<th>Theoretical U/L. w.e.</th>
<th>U/L. w.e.</th>
<th>Theoretical U/L. w.e.</th>
<th>U/L. w.e.</th>
<th>Theoretical U/L. w.e.</th>
<th>U/L. w.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Powell</td>
<td>Slab 6</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>7260</td>
<td>55.0</td>
<td>39.2</td>
<td>37.7</td>
<td>5.4</td>
<td>5.4</td>
<td>5.1</td>
<td>5.1</td>
<td>5.1</td>
</tr>
<tr>
<td></td>
<td>Slab 7</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>8130</td>
<td>35.6</td>
<td>49.4</td>
<td>45.2</td>
<td>5.5</td>
<td>5.5</td>
<td>5.6</td>
<td>5.6</td>
<td>5.6</td>
</tr>
<tr>
<td></td>
<td>Slab 8</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>6750</td>
<td>48.1</td>
<td>40.5</td>
<td>37.3</td>
<td>9.8</td>
<td>9.8</td>
<td>9.8</td>
<td>9.8</td>
<td>9.8</td>
</tr>
<tr>
<td></td>
<td>Slab 9</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>7310</td>
<td>55.9</td>
<td>43.5</td>
<td>39.1</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
</tr>
<tr>
<td></td>
<td>Slab 10</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>6680</td>
<td>55.0</td>
<td>44.2</td>
<td>41.0</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
</tr>
<tr>
<td></td>
<td>Slab 11</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>7200</td>
<td>49.6</td>
<td>53.3</td>
<td>50.0</td>
<td>19.6</td>
<td>19.6</td>
<td>19.6</td>
<td>19.6</td>
<td>19.6</td>
</tr>
<tr>
<td></td>
<td>Slab 12</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>7200</td>
<td>49.6</td>
<td>53.3</td>
<td>50.0</td>
<td>19.6</td>
<td>19.6</td>
<td>19.6</td>
<td>19.6</td>
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</tr>
<tr>
<td></td>
<td>Slab 13</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>7200</td>
<td>50.0</td>
<td>52.6</td>
<td>49.4</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
</tr>
<tr>
<td></td>
<td>Slab 14</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>7200</td>
<td>51.4</td>
<td>51.4</td>
<td>49.4</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
</tr>
<tr>
<td></td>
<td>Slab 15</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>7200</td>
<td>51.4</td>
<td>51.4</td>
<td>49.4</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
</tr>
<tr>
<td></td>
<td>Slab 16</td>
<td>36x0.57x1.266</td>
<td>0.75</td>
<td>0.05 0.05 0.05 0.05 0.05</td>
<td>7200</td>
<td>51.4</td>
<td>51.4</td>
<td>49.4</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
<td>21.9</td>
</tr>
</tbody>
</table>

Note: Mild steel reinforcement was used in all the above tests. Steel % was determined on the basis of the mean effective depths in the case of the slabs of Powell and Wood, and the actual effective depths in the case of the slabs of series A and B.
flexural load which is conservative in most cases. Hence in view of its relative simplicity and its tendency to give a conservative answer it is felt that (2.77) gives a better solution for design purposes than when (2.75) is substituted into (2.76).

2.6.2 Slabs With One Edge Laterally Unrestrained, The Other Edges Fully Restrained.

If the values of the coefficients \( \alpha_x', \alpha_y', \beta_x', \beta_y \) and from (2.20), (2.24), (2.21), (2.25) and (2.22), respectively, the yield moments \( m_x', m_x, m_y \) and \( m_y \) from (2.26), (2.27), (2.28) and (2.29), respectively, and \( \Delta_u = 0.4d \) (as was suggested in section 2.5.2), are substituted into the "approximate" equations, the following expressions for ultimate flexural load (at the point B of Fig. 1.1) result:

**A long edge unrestrained:**

From (2.61), for slabs with \( L_y > L_x \):

\[
\frac{wL_x^2(3\frac{L_y}{L_x} - 1)}{24} = k_1k_3ud^2(0.363 - 0.406k_2) - \frac{k_2}{2k_1k_3u}(T_y'T_y - C_{sy}' + C_{sy})
\]

\[
+ C_{sy}(0.4d - d_y') + C_{sy}'(0.4d - d_y') + T_y(d_y - 0.4d)
\]

\[
+ T_y'(d_y' - 0.4d) + \frac{L_y}{L_x} \left[ T_x(d_x - \frac{k_2T_x}{k_1k_3u}) \right]
\]

\[
+ kT_x'(d_x' - \frac{k_2T_x}{k_1k_3u}) \]

(2.79)
Where in the last term \( k = \frac{1}{2} \) if the laterally unrestrained edge is simply supported, or \( k = 1 \) if the edge is fixed against rotation.

A short edge unrestrained:

From (2.62), for slabs with \( L_y > L_x \):

\[
\frac{w u L_x^2 (3 \frac{L_y}{L_x} - 1)}{24} = k_1 k_3 u d^2 \left\{ \frac{L_y}{L_x} \left( 0.240 - 0.320 k_2 \right) + \left( 0.123 - 0.087 k_2 \right) \right\}
\]

\[
- \frac{k_2}{2 k_1 k_3 u} \frac{L_y}{L_x} \left( T_x' - T_x - C_{sx} + C_{sx} \right)^2
\]

\[
+ C_{sx} \left\{ \frac{d}{10} \left( 3 \frac{L_y}{L_x} + 1 \right) - \frac{L_y}{L_x} d_{x} \right\}
\]

\[
+ C_{sx}' \left\{ \frac{d}{10} \left( 3 \frac{L_y}{L_x} + 1 \right) - \frac{L_y}{L_x} d_{x}' \right\}
\]

\[
+ T_x \left\{ \frac{L_y}{L_x} d_{x} - \frac{d}{10} \left( 3 \frac{L_y}{L_x} + 1 \right) \right\}
\]

\[
+ T_x' \left\{ \frac{L_y}{L_x} d_{x}' - \frac{d}{10} \left( 3 \frac{L_y}{L_x} + 1 \right) \right\} + T_y (d_{y} - \frac{k_2 T_y}{k_1 k_3 u})
\]

\[
+ kT_y (d_{y}' - \frac{k_2 T_y}{k_1 k_3 u}) \tag{2.80}
\]

Where in the last term \( k = \frac{1}{2} \) if the laterally unrestrained edge is simply supported, or \( k = 1 \) if the edge is fixed against rotation.
In (2.79) and (2.80) it has been assumed that the force in the compression steel in the direction in which membrane stresses do not act is zero since it is close to the neutral axis and will contribute little to the strength of the section.

Table 2.4 shows the theoretical ultimate flexural loads calculated from the "exact" (2.45), (2.46), ... (2.60) with $A_u = 0.4d$ substituted, the theoretical ultimate flexural loads calculated from (2.79) and (2.80), the theoretical ultimate loads calculated by Johansen's yield-line theory, and the experimental ultimate flexural loads for the slabs of series B and C. It can be seen that in most cases (2.79) and (2.80) give a good indication of the ultimate flexural strengths of these slabs. The table also illustrates how Johansen's yield line theory underestimates the ultimate loads of these slabs.

2.7 Discussion.

The comparison of theoretical and experimental results in Tables 2.3 and 2.4 shows that on the basis of an empirical value for the central deflection the rigid-plastic strip approximation gives a reasonable estimate of the ultimate flexural loads of slabs loaded to failure in a few hours,
Table 2.4 Tests and Theory: Ultimate Flexural Loads of Slabs with One Edge on Rollers, Other Edges Fully Restrained

<table>
<thead>
<tr>
<th>Authority</th>
<th>Slab Mark</th>
<th>Cube Strength u lb/sq.in.</th>
<th>(a) Experimental ultimate Load wu test lb/sq.in.</th>
<th>(b) Theoretical &quot;Exact&quot; Ultimate Load if ( A_u = 0.4d ) lb/sq.in.</th>
<th>(c) Theoretical &quot;Approx&quot; Ultimate Load ( w_u\text{approx} ) lb/sq.in.</th>
<th>(d) Johnsonen Load ( w_J ) lb/sq.in.</th>
<th>(e) ( w_u\text{approx} ) ( w_u\text{test} )</th>
<th>(f) ( w_u\text{approx} ) ( w_u\text{test} )</th>
<th>(g) ( w_u\text{approx} ) ( w_u\text{test} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Park</td>
<td>B1</td>
<td>4450</td>
<td>20.2</td>
<td>(2.59)(2.60)</td>
<td>(2.80)</td>
<td>7.0</td>
<td>2.89</td>
<td>1.45</td>
<td>1.27</td>
</tr>
<tr>
<td></td>
<td>B2</td>
<td>4520</td>
<td>24.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B3</td>
<td>4680</td>
<td>25.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B4</td>
<td>4720</td>
<td>32.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C1</td>
<td>6250</td>
<td>16.7</td>
<td>(2.51)(2.52)</td>
<td>(2.79)</td>
<td>5.7</td>
<td>2.93</td>
<td>1.43</td>
<td>1.07</td>
</tr>
<tr>
<td></td>
<td>C2</td>
<td>5160</td>
<td>18.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C3</td>
<td>5060</td>
<td>18.1</td>
<td>(2.45)(2.46)</td>
<td>(2.47)</td>
<td>17.9</td>
<td>8.8</td>
<td>1.42</td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td>C4</td>
<td>6360</td>
<td>26.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Slabs B1 and C1 had same dimensions and reinforcement Percentages as slab A1. Similarly alike were A2, B2 and C2, as were A3, B3 and C3, and B4, B4 and C4.
Slab series B had a short edge simply supported.
Slab series C had a long edge simply supported.
the axial shortening of slab elements due to elastic, creep and shrinkage strains in that time evidently having negligible effect in most cases. The slabs of series D which had \( \frac{L}{d} \) ratios of approximately 40 had low experimental ultimate loads, however, and this can be attributed to the effect of axial strains. It will be shown in Chapter 3 that axial strains may have considerable effect on the ultimate flexural loads of thin slabs. The low experimental ultimate loads of two of Wood's slabs cannot be accounted for. One possible explanation is that the reinforced concrete surrounding frame used by Wood may not have applied full lateral restraint.

The general agreement of tests and theory indicates that plastic theory can be applied to unreinforced concrete slabs with restrained boundary conditions. Taking, as the theory has done, the strength of the concrete as that given by uniaxially loaded specimens, instead of the greater biaxial value existing in the slabs, no doubt compensates for the limiting plasticity of concrete in compression, and probably was partly responsible for some of the ultimate loads found in the tests being higher than theory. Analysing a slab as a system of strips may appear to be a crude approximation but it should be noted that a similar strip
approximation for slabs without membrane action gives exactly the same theoretical ultimate flexural strength as the conventional Johansen's yield-line theory, since Johansen's theory considers a yield line to be stepped in the directions of the steel and ignores torsional moments acting on the faces of the steps. For slabs with membrane action, considering two independent systems of strips gives a reasonably accurate idealization when all edges are fully restrained. In the case of slabs with three edges fully restrained and one edge free to translate horizontally, however, a conservative result is given since no membrane forces are considered to be acting at right angles to the laterally unrestrained edge, whereas in the actual slab the continuity between elements will enforce some membrane action in that direction. The vertical deflection profiles obtained from the slabs during the tests are of interest since the theory assumes that the segments of the slabs between the yield lines remain plane. Figure 2.22 shows the vertical deflection profiles measured at the transverse centre lines of the slabs of series A as the load was increased from zero to the ultimate flexural load. It is evident that the assumption of plane segments between the yield lines is fairly accurate at the ultimate flexural load.
POSITION OF DIAL GOUGES.

S = 6.67 in.

FIG. 2.22 VERTICAL DEFLECTIONS ON TRANSVERSE CENTRE LINES OF SLAB SERIES A.
A two-way slab with edges fully restrained against lateral movement may be considered to be the ideal case of an interior panel of a continuous slab and beam floor. A two-way slab with three edges fully restrained and one edge free to translate laterally may be considered to be the ideal case of an edge panel which is surrounded by three panels, or of an interior panel which has a narrow edge panel on one side. The ultimate load equations developed in the theory show that interior panels of slab and beam floors which have surrounding panels which are stiff enough to prevent lateral movement have extremely high ultimate flexural loads, even when they are of the minimum thickness and contain the minimum amount of reinforcement allowed by the Code of Practice. Consider the case of a square slab fully restrained around all edges and reinforced top and bottom with 0.075% of steel to total in each direction the minimum reinforcement content allowed by the code \(^1\) of 0.15% of the gross sectional area of the concrete. Let the yield stress of the steel be 40,000 lb./sq.in., the distance between centroids of top and bottom steel be 0.6 of the slab depth, the cube strength of the concrete be 3000 lb./sq.in. and the span/depth ratio be 40 (the maximum allowed by the code). Then (2.78) gives the ultimate
flexural load of the slab as 746 lb./sq.ft., this load being constant for any span. If a load-factor of 3 is used (a value which is commonly used in reinforced concrete design when the strength is mainly due to the concrete) the working load would be 249 lb./sq.ft., which is in excess of the requirements of most slabs. It is to be noted that in this case Johansen's yield-line theory gives an ultimate load of only about 10% of that given by (2.78), indicating the small contribution the steel makes to the strength.

With such high working loads on such lightly reinforced slabs the cracking and deflection at working load may be thought to be of some concern. The tests showed, however, that the effect of restraint against lateral displacement is to delay the onset of cracking. In the tests on the slabs of series A, B, C and D only the unloaded surface was visible, the other surface being covered by the loading bag. Cracking of the unloaded surface, however, did not become visible until at least 32% of the ultimate flexural load had been applied in the case of the slabs of series B and C, and for the slabs of series A and D this figure was 42%. In fact, in the lightly reinforced slabs cracking did not become visible until the Johansen load had been well exceeded. Hence, it is evident that cracking at working load would be
negligible, and much less than cracking at working load of ordinary reinforced concrete structural elements. Regarding deflections, the load-deflection curves of Figs. 2.17, 2.18, 2.20 and 2.21 show that at 1/3rd of the short-term ultimate flexural load the central deflections were never greater than 1/500 of the short span.
CHAPTER 3

THE ULTIMATE FLEXURAL STRENGTH OF UNIFORMLY LOADED RECTANGULAR
TWO-WAY CONCRETE SLABS INCLUDING THE EFFECT OF AXIAL STRAINS
AND PARTIAL RESTRAINT AGAINST LATERAL DISPLACEMENT AT THE
BOUNDARIES

3.1 Scope of Theory and Tests

The theory for ultimate flexural strength will be
generalized to include the effect of axial strains in the plane
of the slab caused by elasticity, creep and shrinkage of the
concrete. Two-way slabs with either all edges partially
restrained against lateral displacement or with three edges
partially restrained against lateral displacement and one
edge free to translate horizontally will be analysed. The
edges will be considered to be fixed against rotation and
vertical translation, except for the edge free of lateral
restraint which may be simply supported.

The theory will be checked against experimental results
obtained from slabs tested under sustained uniform loading.

3.2 The Effect of Axial Strains and Lateral Displacement
of Edges.

Compressive membrane action in laterally restrained slabs
depends upon the restriction of small outward displacements at
the boundaries and therefore axial shortening of elements of
the slab and partial restraint against lateral displacement at the edges may cause a significant reduction in the ultimate strength. In Chapter 2 the ultimate flexural strength of laterally restrained slabs was found assuming that the slab material was rigid-plastic, and that the boundaries were fully restrained against lateral displacement. Neglect of axial strains in the plane of slabs tested in very stiff surrounding frames was shown to have little effect on the short-term ultimate load, since the creep and shrinkage strains occurring in this time were negligible and the elastic strains alone were too small to have an appreciable effect. If, however, a slab is subjected to a sustained working load for a period of several years before being loaded to failure the creep and shrinkage strains occurring in the concrete may be considerably greater than the elastic strains, and the large total axial strain resulting may cause a considerable reduction in strength. Also the edges may only be partially restrained against lateral displacement if the loaded slab is part of a continuous slab and beam floor in which the lateral restraint is provided by the stiffness of surrounding beams and slabs.

It is of interest to note that in most reinforced concrete structures sustained loading and small boundary displacements
will cause a redistribution of stresses at working load which has little effect on the ultimate strength of the structure. This is not the case for laterally restrained slabs, however.

3.3 Theory for the Ultimate Flexural Strength of Slabs With Axial Strains and Lateral Displacement of Edges.

3.3.1 The Assumptions

The theory will be based upon the strip approximation used in Chapter 2. The assumptions are:

(a) The slab is composed of strips running in the x and y directions (parallel to the slab boundaries) which have the same depth as the slab. The x direction strips contain only the x direction steel and the y direction strips contain only the y direction steel. The external loading on the equivalent slab is shared between the two systems of strips, and the vertical deflections are the same as for the slab.

(b) The yield-line patterns of slabs with the boundary conditions considered are as shown in Fig. 3.3. (The accuracy of these approximate yield-line patterns has been discussed in Chapter 2). The yield sections of the strips lie on the yield lines of the actual slab.

(c) The yield sections of the strips are at right angles to the directions of the strips. At the yield sections the torsional moments are zero, the tension steel has yielded
and the compressed concrete has reached its ultimate value. The strength of concrete in tension is ignored.

(d) The portions of the strips between the yield sections remain straight.

(e) The top steel at opposite supports is the same but may be different from the top steel in the other direction. The bottom steel is placed over the whole area of the slab. The steel area, per unit width, is constant for each layer of steel, but may be different for steel in each direction and for top and bottom steel.

(f) The sum of the elastic, creep and shrinkage axial strains in each strip is the same for all strips running in the same direction, but may be different for x and y direction strips.

(g) The outward lateral displacement which occurs at each boundary is the same for strips running in the same direction, but may be different for x and y direction strips.

(h) The slab reaches its ultimate flexural strength at a central deflection of either 0.5 of the slab thickness if all edges are laterally restrained, or 0.4 of the slab thickness if three edges are laterally restrained and one edge is laterally unrestrained. (These are conservative values which were found in Chapter 2 from the short-term loading tests).
Let the strip form the general collapse mechanism shown in Fig. 3.1 at the ultimate flexural load. The changes in length of the portions of the strip due to axial strain will now be examined. The axial strain in the strip, \( \varepsilon \) (the sum of the elastic, creep and shrinkage strains), will have a constant value at any point along the length of the strip since the membrane force is constant. Due to \( \varepsilon \), the shortening of the middle portion 23 of the strip will be \( \varepsilon (1-2\rho) L \) and, because of symmetry, the ends of the portion 23 will approach the centre of the strip by \( 0.5 \varepsilon (1-2\rho) L \). The outward lateral displacement of each boundary due to partial restraint is \( t \). Hence the horizontal distance from each end of the portion 23 of the strip to the adjacent boundary is \( \rho L + 0.5 \varepsilon (1-2\rho) L + t \). Also, due to \( \varepsilon \) the lengths of end portions 12 and 34 will decrease by \( \varepsilon \rho L \) to \( (1-\varepsilon) \rho L \). Fig. 3.2 shows the end portion 12 of the strip and the changes in dimensions due to \( \varepsilon \) and \( t \).

The distance between points A and C of the figure is:

\[
\{\rho L + 0.5 \varepsilon (1-2\rho) L + t\} \sec \phi = AB + (1-\varepsilon) \rho L - CD
\]

\[
= (1-n_1') d \tan \phi + (1-\varepsilon) \rho L
\]

\[
- n_1 d \tan \phi
\]

\[
\therefore 1-n_1'-n_1 = \frac{2 \rho L \sin^2 \theta + 0.5 \varepsilon \rho L \cos \phi + 0.5 \varepsilon (1-2\rho) L + t}{d \sin \phi}
\]
FIG. 3.1 COLLAPSE MECHANISM OF STRIP WITH PARTIAL RESTRAINT AT ENDS.

FIG. 3.2 INTERNAL ACTIONS AT YIELD SECTIONS OF END PORTION OF STRIP.
But since $\phi$ is small:

$$\sin \phi = 2 \sin 0.5\phi = \frac{z}{\rho L}, \text{and } \cos \phi = 1.$$ 

Hence

$$n'_1 + n_1 = 1 - \frac{z}{2d} - \frac{\rho L^2}{2dz} (\varepsilon + \frac{2t}{L})$$  \hspace{1cm} (3.1)

Also for equilibrium of horizontal forces acting on portion 12 of the strip:

$$C'_c + C'_s - T' = C_c + C_s - T$$

$$n'_1 - n_1 = \frac{T' - T - C'_s + C_s}{k_1 k_3 u d}$$  \hspace{1cm} (3.2)

On solving (3.1) and (3.2) simultaneously, and putting

$$\varepsilon' = (\varepsilon + \frac{2t}{L})$$  \hspace{1cm} (3.3)

the following equations result for $n'_1$ and $n_1$:

$$n'_1 = 0.5 \left(1 - \frac{z}{2d} - \frac{\varepsilon' \rho L^2}{2dz} + \frac{T' - T - C'_s + C_s}{k_1 k_3 u d}\right)$$  \hspace{1cm} (3.4)

$$n_1 = 0.5 \left(1 - \frac{z}{2d} - \frac{\varepsilon' \rho L^2}{2dz} - \frac{T' - T - C'_s + C_s}{k_1 k_3 u d}\right)$$  \hspace{1cm} (3.5)
Comparison of (3.4) and (3.5) with (2.8) and (2.9) shows that the effect of axial shortening of strips and lateral displacements at the boundaries is to reduce the depths to the neutral axes at the yield sections. The actions at the yield sections of the strips, \( N \) (constant along the length), \( M \) and \( M' \), are given by (2.1), (2.2) and (2.4) as before.

Considering the end portion 12 or 34 of the strip, the sum of the moments of the actions at the yield sections about one end is \( M' + M - N_z \). Shear forces have been neglected since their net contribution to the analysis by virtual work will be zero. On substituting \( n_i' \) and \( n_i \) from (3.4) and (3.5) into the equations for \( N, M \) and \( M' \), (2.1), (2.2) and (2.4), it is found that:

\[
M' + M - N_z = k_1 k_3 u d \left\{ \frac{1}{2} d(1-k_2) + \frac{1}{4} z(2k_2-3) + \frac{\epsilon' \rho L^2}{4z} (2k_2-1) \right. \\
+ \frac{z^2}{8d} (2-k_2) + \frac{\epsilon' \rho L^2}{4d} (1-k_2) - \frac{k_2 (\epsilon')^2 \rho^2 L^4}{8dz} \left. \right\} \\
- \frac{k_2}{2k_1 k_3 u} (T' - T - C_s' + C_s)^2 + C_s' \left( \frac{1}{2} d - d_i' - \frac{1}{2} z \right) \\
+ C_s \left( \frac{1}{2} d - d_2 - \frac{1}{2} z \right) + T' \left( d_i' - \frac{1}{2} d + \frac{1}{2} z \right) + T \left( d_i - \frac{1}{2} d + \frac{1}{2} z \right) \\
\]  

(3.6)

It will be noticed that (3.6) is dependent upon \( \rho L \) (the lengths of the end portion of strips) as well as \( z \) (the vertical
derlection of the interior yield sections) and the slab properties. In the case of the rigid-plastic strip dealt with in Chapter 2, \( M', M \) and \( N \) were all independent of \( pL \). If the portion 12 of the strip is given a virtual rotation \( \Theta \) by giving the centre of the strip a virtual displacement in the direction of the loading, the virtual work done by the internal actions at the yield sections is:

\[
- (M' + M - Nz)\Theta, \text{ per unit width.} \quad (3.7)
\]

Also, if \( N = 0 \) the membrane stresses are zero and the moments \( M' \) and \( M \) become the yield moments without membrane stresses, \( m' \) and \( m \), given by (2.17) and (2.18).

3.3.3 Uniformly Loaded Two-Way Rectangular Slab With All Edges Fully Fixed Against Rotation and Vertical Translation But Partially Restrained Against Lateral Translation.

Figure 3.3a shows the assumed yield-line pattern. The deflection at the yield-line EF at the ultimate flexural load is assumed to be 0.5d. Hence if the origin of coordinates is taken at A, at any point \((x,y)\) on the yield line AE:

\[
z = \frac{dx}{L_x} \quad (3.8)
\]

or \[
z = \frac{dy}{L_x} \quad (3.9)
\]
Hogging moment yield line
Sagging moment yield line
Fixed edge
Simply supported edge

FIG. 3.3 APPROXIMATE YIELD-LINE PATTERNS FOR UNIFORMLY LOADED TWO-WAY SLABS.
The deflections at points on the other yield lines DE, BF and CF follow from the above by symmetry.

Considering the slab to be composed of systems of strips in the x and y directions, it can be seen that the values of \( \rho \) (defining the lengths of the end portions of strips as a proportion of the spans of the strips) vary between 0 and 0.5.

For the x direction strips intersecting line AE,
\[
\rho = \frac{X}{L_x} \quad (3.10)
\]

For the x direction strips intersecting line EF,
\[
\rho = 0.5 \quad (3.11)
\]

For y direction strips intersecting line AE,
\[
\rho = \frac{X}{L_y} \quad (3.12)
\]

If at the ultimate flexural load the equivalent slab is given a virtual displacement \( \delta \) at the yield line EF in the direction of the loading, the slab segments, made up of portions of strips, will undergo virtual rotations of \( \frac{2\delta}{L_x} \) about the lines of yielding at the edges of the slab. It will be noticed that only the end portions of the strips will undergo rotations, the middle portion of those strips divided into three portions remaining horizontal. Hence the virtual work done by the actions at the yield sections will be given by (3.7) written for the end portions of x and y
direction strips and summed over the whole slab.

Virtual work done by the x direction strips (due to those strips in segments ABFE and DCFE only) is, from (3.7):

\[
\mathbf{f} \cdot 5L_x (M'_x + M_x - N_x z) \frac{25}{L_x} dy - 2(M'_x + M_x - N_x z) \frac{25}{L_x} (L_y - L_x)
\]

(3.13)

The first term represents contributions from those strips intersecting yield lines AE, DE, BF and CF, and the second term represents contributions from those strips intersecting yield line EF. The values of \( L'_x + M_x - N_x z \) for (3.13) may be obtained by substituting into (3.6) the appropriate x direction quantities and also \( z \) from (3.9) and \( \rho \) from (3.10) into the first term, and \( z = 0.5d \) and \( \rho \) from (3.11) into the second term. On performing these substitutions and on integrating and rearranging the terms of (3.13), it is found that:

Virtual work done by the x direction strips

\[
= - \delta k_1 k_3 u d^2 \left[ \left( \frac{7-5k_2}{12} + \frac{L_y}{L_x} \frac{3}{8} (2-3k_2) \right) + \frac{\varepsilon'_X L^2}{4d^4} (k_2 - 1) + \frac{L_y}{L_x} (3k_2 - 1) \right] \\
- \frac{k_2}{2} (\varepsilon'_X)^2 \frac{L_y}{L_x} \left( \frac{L}{d} \right) - \delta \left[ C_{sx} \left\{ \frac{L_y}{L_x} (d - 4d'_2 x) \right\} + \frac{d}{2} \right] \\
+ C_{sx} \left[ \frac{L_y}{L_x} (d - 4d'_2 x) + \frac{d}{2} \right] + T'_x \left[ \frac{L_y}{L_x} (4d'_1 x - d) - \frac{d}{2} \right]
\]
Also the virtual work done by the \( y \) direction strips (due to those strips in segments ADE and BCF only) is, from (3.14):

\[
= -4 \int_0^{0.5L} (M_y + M_y - N_y z) \frac{2L}{L_x} \, dx
\]  

(3.15)

The value of \( M_y + M_y - N_y z \) required for (3.15) may be obtained by substituting into (3.6) the appropriate \( y \) direction quantities and also \( z \) from (3.8) and \( \rho \) from (3.12). On performing these substitutions and on integrating and rearranging the terms of (3.15), it is found that:

Virtual work done by the \( y \) direction strips

\[
= -\delta k_3 u d^2 \left\{ \frac{32 - 37k_2}{24} + \frac{\epsilon' L^2 L}{4a^2 L_x} (7k_2 - 3) - \frac{i k_2}{2} (\epsilon'_y)^2 \left( \frac{L_y}{L_x} \right)^2 \left( \frac{L}{d} \right)^4 \right\}
- \delta \left[ C_{sy} \left( \frac{3}{2} d - 4a_2 y \right) + C_{sy} \left( \frac{3}{2} d - 4a_2 y \right) + T_y (4a_1 y - \frac{3}{2} d) 
+ T_y (4a_1 y - \frac{3}{2} d) \right] - \frac{2k_2}{\delta k_3 u} (T_y - T_y - C'_sy + C_{sy})^2 
\]  

(3.16)

The virtual work done by the ultimate load, \( w_u \), is:

\[
= w_u \left\{ \frac{L_x \delta}{3} + (L_y - L_x) L_x \frac{\delta}{2} \right\}
\]

\[
= \frac{w_u L^2}{6} \left( 3 \frac{L_y}{L_x} - 1 \right)
\]  

(3.17)
Now from the virtual work equation:

\[(3.14) + (3.16) + (3.17) = 0.\]

Hence the following equation for ultimate strength of slabs with \(L_y > L_x\) is obtained:

\[
w_u L_x^2 \left(3 \frac{L_y}{L_x} - 1\right) = k_1 k_3 u d^2 \left(\frac{1}{4}(46 - 47k_2) + \frac{9}{4} \frac{L_y}{L_x} (2 - 3k_2) \right.\]
\[+ \frac{3}{2} \varepsilon_x' \left(\frac{L_y}{d}\right)^2 \left[\left(k_2 - 1\right) + \frac{2L_y}{L_x} (3k_2 - 1)\right] \]
\[+ \frac{3}{2} \varepsilon_y' \left(\frac{L_y}{d}\right)^2 \frac{L_y}{L_x} (7k_2 - 3) - 3k_2 \frac{L_y}{L_x} \frac{L_y}{d} \left[\left(e_x'^2\right)^2 \right.\]
\[\left. + \frac{L_y}{L_x} \left(e_y'^2\right)^2\right]\left[2T_x \left(C_{sx'} + C_{sz'}\right)^2 + \left(T_{y'} - T_y \cdot C_{sy} + C_{sy}\right)^2\right] + 24 \left[C_{sx} \left[\frac{L_y}{L_x} \left(\frac{d}{4} - d_x'\right) + \frac{d}{8}\right] + T_x \left[\frac{L_y}{L_x} \left(d_x' - \frac{d}{4}\right) - \frac{d}{8}\right]\right.\]
\[\left. + T_x \left[\frac{L_y}{L_x} \left(d_y' - \frac{d}{4}\right) - \frac{d}{8}\right] + C_{sy} \left(\frac{3d}{8} - d_y'\right)\right] + C_{sy} \left(\frac{3d}{8} - d_y'\right) + T_y' (d_y' - \frac{3d}{8}) + T_y (d_y - \frac{3d}{8}) \right]

(3.18)

where the values of \(\varepsilon_x'\) and \(\varepsilon_y'\) are from (3.3)

\[
\varepsilon_x' = (\varepsilon_x + \frac{2t_{ux}}{L_x})
\]

(3.19)

\[
\varepsilon_y' = (\varepsilon_y + \frac{2t_{uy}}{L_y})
\]

(3.20)
Examination of the equations defining the depths to the neutral axes, (3.4) and (3.5), shows that it is theoretically possible for the neutral axis depths at the yield sections to be reduced to zero at the ultimate load of slabs with high span/depth ratios and high $\varepsilon'$ values. If this occurs the strips will carry the external load by steel couples only, the concrete making no contribution to the strength of the slab. Unfortunately (3.18) does not apply to this situation. Once the neutral axis is outside the section, (3.18) makes the contribution of the concrete negative instead of zero and the sign of the force of what was the compression steel is not reversed. Hence the equations have to be rewritten for slabs with high $\varepsilon'$ values and span/depth ratios.

For the $x$ direction strips of slabs with equal top and bottom steel, (3.4) and (3.5) show that:

$$ n_1' = n_1 = 0.5(1 - \frac{z}{2d} - \frac{\varepsilon'_x \rho L^2}{2dz}) $$

The above equation is also approximately true of slabs with unequal top and bottom steel since the steel forces have only a small influence on the neutral axis depths. Reference to Fig.3.3 illustrates that with the origin at A and with a deflection at E of 0.5d, for an $x$ direction strip passing through the yield line AE the variables $\rho$ and $z$ may be
written as:

\[ \rho = \frac{V}{L_x} \quad z = \frac{vd}{L_x} \]

Hence for the x direction strips passing through the yield line AE:

\[ n'_x = n_x = 0.5 \left(1 - \frac{V}{2L_x} - \frac{\varepsilon_x' L_x^2}{2d^2} \right) \]

\[ \therefore \quad n'_x = n_x = 0 \]

when \( \varepsilon_x' = \frac{2d^2}{L_x^2} \left(1 - \frac{V}{2L_x} \right) \) \hspace{1cm} (3.21)

Now the minimum value of \( \varepsilon_x' \) to cause \( n'_x = n_x = 0 \) is sought. (3.21) shows that \( \varepsilon_x' \) will be a minimum when \( \frac{L_x}{d} \) is a maximum. If \( \frac{L_x}{d} = 40 \) (the maximum value allowed by the Code of Practice) is considered, then:

At \( y = 0 \) (at the slab corners) the neutral axis depth is zero when

\[ \varepsilon_x' = 2 \times \left(\frac{1}{40}\right)^2 = 1250 \times 10^{-6} \]

At \( y = \frac{L_x}{2} \) (at point E and along yield line EF) the neutral axis depth is zero when

\[ \varepsilon_x' = 2 \times \left(\frac{1}{40}\right)^2 \times 0.75 = 940 \times 10^{-6} \]
These two values of $\varepsilon'_x$ indicate the limits of the transition range during which the neutral axis depths at the yield sections of the $x$ direction strips become zero over the whole of the slab, starting in the central region. It is unlikely that $\varepsilon'_x$ values as high as these will be reached in practice, however.

In the case of the $y$ direction strips it can be shown by reasoning similar to that leading to (3.21) that at the yield sections:

$$n'_1 = n_1 = 0$$

when

$$\varepsilon'_y = \frac{2L^2}{L_x L_y} \frac{L_x}{L_y} (1 - \frac{x}{2L_x})$$

Hence in this case smaller strains will cause the neutral axis depths to be reduced to zero, since $L_y > L_x$. For example, if $\frac{L_y}{L_x} = 3$ and $\frac{L_x}{d} = 40$, a value of $\varepsilon'_y = 310 \times 10^{-6}$ at the slab centre would be sufficient. An estimate of the ultimate load of the slab for this situation can be made by putting the contribution of the concrete in the $y$ direction strips equal to zero in (3.16) and rewriting the virtual work equation. Then it may be shown that with only the concrete in the $x$ direction strips and the steel in the $x$ and $y$ direction strips carrying load, for slabs with $L_y > L_x$: 
In (3.23) the contribution of what was the y direction compression steel has been put equal to zero since although it will now be in tension its contribution to the strength of the slab will be small. (3.22) defines the limits of the ranges of applicability of (3.18) and (3.23). Strictly, (3.18) governs before the neutral axis depths start to become zero and (3.23) governs after all the neutral axis depths in the y direction strips have become zero. Hence as for the case of the x direction strips, there is a transition range during which as \( \varepsilon_y' \) increases the neutral axis depths are gradually becoming zero. Thus a further equation for ultimate flexural load should really be written for this transition range but, considering the accuracy of
the method, such a further refinement is unwarranted. It is suggested that (3.16) and (3.23) should each be considered to be applicable over their adjacent half of the transition range. This is a conservative assumption. At the centre of the transition range \( x = \frac{L_x}{4} \), and hence from (3.22):

\[
\text{If } \varepsilon_y' < 1.75 \left( \frac{d}{L_x} \right)^2 \frac{L_x}{L_y}, (3.18) \text{governs,} \quad \]

\[
\text{and if } \varepsilon_y' > 1.75 \left( \frac{d}{L_x} \right)^2 \frac{L_x}{L_y}, (3.23) \text{governs,} \quad \]

where \( \varepsilon_x' \) and \( \varepsilon_y' \) are given by (3.19) and (3.20).

The possibility of the neutral axis depths in the \( x \) direction strips being reduced to zero has not been included since, as (3.21) showed, very high \( \varepsilon_x' \) values would be required.

3.3.4 Uniformly Loaded Two-Way Rectangular Slab With Three Edges Fully Fixed Against Rotation and Vertical Translation But Partially Restrained Against Lateral Translation, and the Remaining Edge Laterally Unrestrained.

Figure 3.3b and c show the assumed yield-line patterns for the cases of a long edge and a short edge laterally unrestrained and simply supported. If the laterally unrestrained edge had been fixed against rotation each yield-line pattern would be as shown with the addition of a hogging moment yield line along the laterally
unrestrained edge. The deflection at each yield line EF at the ultimate flexural load is assumed to be \(0.4d\). Hence if the origin of coordinates is taken at A, at any point \((x,y)\) on the yield lines AE:

\[
z = 0.8 \frac{xd}{L_x}, \tag{3.25}
\]

or \(z = 0.8 \frac{yd}{L_x}\). \(\tag{3.26}\)

The deflections at the points on the other yield lines, DE, BF and CF, follow from the above by symmetry. Considering the slabs to be made up of systems of strips in the x and y directions it can be seen that the values of \(\rho\), defining the lengths of the end portions of the strips as a proportion of the spans of the strips, vary between 0 and 0.5.

For x direction strips intersecting line AE, \(\rho = \frac{y}{L_x}\). \(\tag{3.27}\)

For x direction strips intersecting line EF, \(\rho = 0.5\). \(\tag{3.28}\)

For y direction strips intersecting line AE, \(\rho = \frac{x}{L_y}\). \(\tag{3.29}\)
(a) **A Long Edge Unrestrained Laterally**

If at the ultimate flexural load the equivalent slab with the collapse mechanism shown in Fig. 3.3b is given a virtual displacement $\delta$ at the yield line EF in the direction of the loading, the slab segments, made up of portions of strips, will undergo virtual rotations of $\frac{2\delta}{L_x}$ about the edges of the slab. As in previous cases only the end portions of strips will do virtual work. Because of the edge condition at AB there will be no membrane force in the $x$ direction strips ($N_x=0$), and therefore the actions at yield sections in the $x$ direction strips are the Johansen yield moments $m_x'$ and $m_x$. In the $y$ direction strips membrane forces do exist and the actions at yield sections are $M_y'$, $M_y$ and $N_y$.

Virtual work done by $x$ direction strips (due to those strips in segments ABFE and DCFE only) is:

$$
= - m_x^* \frac{2\delta}{L_x} L_y - (m_x' + m_x) \frac{2\delta}{L_x} L_y
$$

(3.30)

Where $m_x^* = m_x$ if the laterally unrestrained edge is simply supported, or $m_x^* = m_x' + m_x$ if the edge is fixed against rotation. $m_x'$ and $m_x$ are given by (2.26) and (2.27).
Virtual work done by $y$ direction strips (due to those strips in segments $ADE$ and $BCF$ only) is, from \( \ref{eq:2.14} \):

\[
-4 \int_0^{0.5L} \left( M_y' + M_y - N_y z \right) \frac{2\delta}{L_x} \, dx
\]

The value of $M_y' + M_y - N_y z$ required for \( \ref{eq:3.31} \) may be obtained by substituting into \( \ref{eq:3.6} \) the appropriate $y$ direction quantities and also $z$ from \( \ref{eq:3.25} \) and $\rho$ from \( \ref{eq:3.29} \). On performing these substitutions and on integrating and rearranging the terms of \( \ref{eq:3.31} \), it is found that:

Virtual work done by the $y$ direction strips

\[
= -\delta k_1 k_3 u d^2 \left\{ \frac{109-122k_2}{75} + \frac{e_y'}{4} \frac{L_x L_y}{d^2} (9k_2 - 4) - \frac{25}{32} (e_y') k_2 \frac{L_x^2 L_y^2}{d^4} \right\} \\
- \delta \left[ C_{sy}' (1.6d - 4d_2') + C_{sy} (1.6d - 4d_2') + T_y' (4d_2' - 1.6d) \right. \\
+ T_y (4d_2 - 1.6d) - \frac{2k_2}{k_1 k_3 u} \left( T_y' - T_y - C_{sy}' + C_{sy} \right)^2 \right\} \quad \text{(3.32)}
\]

Also, virtual work done by the ultimate load, $w_u$, is

\[
= w_u \left\{ I_x \frac{\delta}{3} + (L_y - L_x) L_x \frac{\delta}{2} \right\} \\
= \frac{w_u \delta L_x}{6} \left( 3 \frac{L_y}{L_x} - 1 \right) \quad \text{(3.33)}
\]
From the virtual work equation (3.30)+(3.32)+(3.33) = 0.
From which the following equation for the ultimate strength of slabs with $L_y > L_x$ is obtained:

$$w_u L_x^2 \left( \frac{L_y}{L_x} - 1 \right) \frac{L_y}{24} = k_1 k_3 u d^2 \left\{ \left( 0.363 - 0.406 k_2 \right) + \epsilon'_y \left( \frac{x}{d} \right)^2 \frac{L_y}{L_x} \left( 0.562 k_2 - 0.25 \right) 
- 0.195 \epsilon'_y^2 k_4 \left( \frac{L_y}{L_x} \right)^2 \left( \frac{x}{d} \right)^4 \right\} 
+ C_{sy} (0.4 d - d'_1 y) + C_{sy} (0.4 d - d_2 y) + T'_y (d'_1 y - 0.4 d) 
+ T_y (d'_1 y - 0.4 d) - \frac{k_2}{2 k_1 k_3 u} \left( T'_y - T_y - C'_{sy} + C_{sy} \right)^2 
+ \frac{L_y}{L_x} \left[ k T'_x \left( d'_1 x - \frac{k_2 T'_x}{k_1 k_3 u} \right) + T_x \left( d'_1 x - \frac{k_2 T_x}{k_1 k_3 u} \right) \right] \tag{3.34}$$

Where in the last term $k = \frac{1}{2}$ if the laterally unrestrained edge is simply supported, or $k = 1$ if the edge is fixed against rotation.

As has been shown for the case for all edges fully restrained in Section 3.3.3, if $\epsilon'_y$ and the $\frac{L_y}{d}$ ratio are large the neutral axis depths at the yield sections of the $y$ direction strips may be reduced to zero at the ultimate flexural load. Then in the $y$ direction strips the load is carried by the steel couples only. Putting the contribution of the concrete in the $y$ direction strips equal to zero in
(3.34), for slabs with $L_y > L_x$ the ultimate load expression becomes:

$$\frac{w_u L^2 (L_y L_x - 1)}{24} = T_y(d'_y - 0.4d) + T_y(d'_y - 0.4d)$$

$$+ \frac{L_y}{L_x} \left( k'_x(d'_x - \frac{k_2 T'_x}{k_1 k_3 u}) + T_x(d'_x - \frac{k_2 T'_x}{k_1 k_3 u}) \right)$$

(3.35)

Where in the last term $k = \frac{1}{2}$ if the laterally unrestrained edge is simply supported, or $k = 1$ if the edge is fixed against rotation. In (3.35) the contribution of what was the $y$ direction compression steel has been put equal to zero since, although it will now be in tension, its contribution to the strength of the slab will be small.

As in Section 3.3.3, (3.24) indicates the range over which (3.34) and (3.35) are applicable.

If $\varepsilon'_y < 1.75 \left( \frac{d}{L_x} \right)^2 \frac{L_x}{L_y}$, (3.34) governs,

and if $\varepsilon'_y > 1.75 \left( \frac{d}{L_x} \right)^2 \frac{L_x}{L_y}$, (3.35) governs.

(3.36)
(b) **A Short Edge Unrestrained Laterally**

If at the ultimate flexural load the equivalent slab with the collapse mechanism shown in Fig. 3.3c is given a virtual displacement \( \delta \) at the yield line EF in the direction of the loading, the slab segments, made up of portions of strips, will undergo virtual rotations of \( \frac{2\delta}{L_x} \) about the edges of the slab. As in the previous cases only the end portions of strips will do virtual work. Because of the edge condition at BC there will be no membrane force in the y direction strips \( (N_y = 0) \), and therefore the actions at the yield sections in the y direction strips are the Johansen yield moments \( m'_y \) and \( m_y \). In the x direction strips membrane forces do exist and the actions at the yield sections are \( M'_x \), \( M_x \) and \( N_x \).

Virtual work done by the x direction strips (due to those strips in segments ABFE and DCFE only) is, from (3.37):

\[
-4 \int_0^{0.5L_x} (M'_x + M_x - N_x z) \frac{2\delta}{L_x} \, dy - 2(M'_x + M_x - N_x z) \frac{2\delta}{L_x} (L_y - L_x) \quad (3.37)
\]

The values of \( M'_x + M_x - N_x z \) for (3.37) may be obtained by substituting into (3.6) the appropriate x direction quantities and also \( z \) from (3.26) and \( \rho \) from (3.27) into the first term, and \( z = 0.4d \) and \( \rho \) from (3.28) into the second term. On
performing these substitutions and integrating and rearranging the terms of (3.37), it is found that:

Virtual work done by the x direction strips

\[
\begin{align*}
&= -\delta k_1 k_3 ud^2 \left\{ \frac{37 - 26k_2}{75} + \frac{8}{25} \frac{L_y}{L_x} (3 - 4k_2) + \frac{\varepsilon_x' L_x}{L_y} (\frac{L_x}{d})^2 \left[ (k_2 - 1) + \frac{L_y}{L_x} (8k_2 - 3) \right] \\
&\quad - \frac{25}{32} (\varepsilon_x')^2 k_2 \left( \frac{L_x}{d} \right)^4 \frac{L_y}{L_x} \right\} \\
&\quad - \delta \left\{ C_{sx}' \left[ 0.4d + \frac{L_y}{L_x} (1.2d - 4d_2') \right] + C_{sx} \left[ 0.4d + \frac{L_y}{L_x} (1.2d - 4d_2'') \right] \\
&\quad + T_x' \left[ \frac{L_y}{L_x} (4d_1' - 1.2d) - 0.4d \right] + T_x \frac{L_y}{L_x} (4d_1'' - 1.2d) - 0.4d \right\} \\
&\quad - \frac{2k_2}{k_1 k_3 u} \frac{L_y}{L_x} (T_x' - T_x - C_{sx}' + C_{sx})^2 \right\} \\
&= -m_y^* \frac{2\delta}{L_x} \frac{L_x}{L_y} - (m_y' + m_y) \frac{2\delta}{L_x} \frac{L_x}{L_y} \quad (3.38)
\end{align*}
\]

Also, the virtual work done by the y direction strips (due to those strips in segments ADE and BCF only) is:

\[
\begin{align*}
&= -m_y^* \frac{2\delta}{L_y} L_x - (m_y' + m_y) \frac{2\delta}{L_x} L_x \quad (3.39)
\end{align*}
\]

Where \(m_y^* = m_y\) if the laterally unrestrained edge is simply supported, or \(m_y^* = m_y' + m_y\) if the edge is fixed against rotation.

\(m_y'\) and \(m_y\) are given by (2.28) and (2.29).
Also the virtual work done by the ultimate load, $w_u$, is

$$w_u = \frac{w_u}{6} \left( 3 \frac{L_y}{L_x} - 1 \right)$$  \hspace{1cm} (3.40)

From the virtual work equation $(3.38) + (3.39) + (3.40) = 0.$

From which the following equation for the ultimate flexural strength of slabs with $L_y = L_x$ is obtained:

$$w_u \frac{L_x^2 (3 \frac{L_y}{L_x} - 1)}{24} = k_1 k_3 u d^2 \left\{ (0.123-0.087k_2) + \frac{L_y}{L_x} (0.240-0.320k_2) \\
+ \varepsilon_x \left( \frac{L_x}{d} \right)^2 \left[ 0.063(k_2-1) + \frac{L_y}{L_x} (0.5k_2-0.188) \right] \\
- 0.195(\varepsilon_x')^2 k_2 \left( \frac{L_x}{d} \right)^4 \frac{L_y}{L_x} \right\} \\
+ C'_{sx} \left[ \frac{L_y}{L_x} (0.3d-d_{x}^e) + 0.1d \right] + C_{sx} \left[ \frac{L_y}{L_x} (0.3d-d_{x}^e) + 0.1d \right] \\
+ T_x' \left[ \frac{L_y}{L_x} (d_{x}^e - 0.3d) - 0.1d \right] + T_x \left[ \frac{L_y}{L_x} (d_{x}^e - 0.3d) - 0.1d \right] \\
- \frac{k_2}{2k_1 k_3 u} \frac{L_y}{L_x} (T_x' - T_x) - C'_{sx} + C_{sx})^2 \\
+ T_y (d_{y}^e - \frac{k_2 T_y}{k_1 k_3 u}) + k T_y (d_{y}^e - \frac{k_2 T_y}{k_1 k_3 u}) \hspace{1cm} (3.41)$$

Where in the last term $k = \frac{1}{2}$ if the laterally unrestrained edge is simply supported, or $k = 1$ if the edge is fixed against rotation.
As has been shown in Section 3.3.3 for the case of all edges fully fixed, the neutral axis depths at yield sections at ultimate load will not be reduced to zero in practical cases.

3.4 The Magnitude of the Membrane Forces

In order to estimate the magnitudes of $\varepsilon_x'$ and $\varepsilon_y'$ for the ultimate load equations the magnitudes of the membrane forces must be determined. The axial strains in the plane of the slab will depend upon the magnitudes of the membrane stresses at mid-depth of the strips, and the horizontal displacements of the boundaries will depend upon the magnitudes of the membrane forces per unit width acting on the boundaries.

The membrane force per unit width is given by substituting (3.5) in (2.1). The following expression results:

$$N = k_1k_3ud\left(\frac{1}{2} - \frac{Z}{4d} - \frac{\varepsilon_0'\rho L^2}{4dz}\right) + \frac{1}{2}(C_s' + C_s - T' - T)$$

N is constant along the length of each strip but varies from strip to strip since $z$ and $\rho$ are variables depending upon the position of the strip in the yield-line pattern. It has been assumed in the theory, however, that the axial
strains and lateral displacements at the boundaries have constant values for all strips in the same direction (but may differ in the x and y directions). These strains and displacements are very difficult to estimate accurately and refined efforts to include their exact variation across the slab in each direction are unwarranted in view of the uncertainty with which some of the quantities have to be estimated. Hence it will be assumed that the effect of axial strains and lateral boundary displacements can be estimated with sufficient accuracy using the mean membrane force per unit width in each direction. The mean value of the membrane force per unit width in a particular direction may be determined by finding the total membrane force in that direction and dividing it by the total width of the strips along which it acts.

For x direction strips:

Referring to Fig. 3.3, if the origin of coordinates is at point A:

For $0 < y < 0.5L_x$, $z = \frac{2yA}{L_x}$ and $\rho = \frac{y}{L_x}$.

For $0.5L_x < y < L_y - 0.5L_x$, $z = \Delta$ and $\rho = 0.5$. 
The mean x direction membrane force per unit width at ultimate load is, from (3.42):

\[
\text{Mean } N_x = \frac{1}{L_y} \left\{ 2 \int_0^{0.5L_x} xN_x \, dy + N_x (L_y - L_x) \right\}
\]

\[
= \frac{1}{L_y} \left\{ 2k_1 k_3 ud \int_0^{0.5L_x} x \frac{1}{2} - \frac{vA}{2dL_x} - \frac{x^2}{8dA} \right\} dy
\]

\[
+ \frac{L_x}{2} (C_{sx} + C_{sx} - T'_x - T_x) + k_1 k_3 ud \left( \frac{1}{2} - \frac{A}{4d} - \frac{\varepsilon'_x L_x^2}{8dA} \right) (L_y - L_x)
\]

\[
= k_1 k_3 ud \left( \frac{1}{2} + \frac{\Delta L_x}{8dL_y} - \frac{\Delta}{4d} - \frac{\varepsilon'_x L_x^2}{8dA} \right) + \frac{1}{2} (C'_{sx} + C_{sx} - T'_x - T_x).
\]

(3.43)

For y direction strips:

Referring to Fig. 3.3, if the origin of coordinates is at point A:

For \(0 < x < 0.5L_x\), \(z = \frac{2x \Delta}{L_x}\) and \(\rho = \frac{x}{L_y}\)

The mean y direction membrane force per unit width at ultimate load is, from (3.42):

\[
\text{Mean } N_y = \frac{2}{L_x} \int_0^{0.5L_x} x N_y \, dx
\]
For slabs with all edges fully fixed it is assumed that
\( \Delta = 0.5d \). Then from (3.43) and (3.44) at ultimate load:

\[
\begin{align*}
\text{Mean } N_x &= k_1 k_3 u d \left[ \frac{3}{8} + \frac{1}{16} \frac{L_x}{L_y} - \frac{\varepsilon'_L X^2}{4} \right] + \frac{1}{2} (C'_s + C_s - T'_y y - T_y) \\
\text{Mean } N_y &= k_1 k_3 u d \left[ \frac{7}{16} - \frac{\varepsilon'_L Y^2}{4} \right] + \frac{1}{2} (C'_s + C_s - T'_y y - T_y)
\end{align*}
\] (3.45, 3.46)

For slabs with one edge laterally unrestrained, the other edges fully fixed, it is assumed that \( \Delta = 0.4d \). From (3.43) and (3.44) for the fully restrained spans at ultimate load:

\[
\begin{align*}
\text{Mean } N_x &= k_1 k_3 u d \left[ 0.4 + 0.45 \frac{L_x}{L_y} - 0.312 \varepsilon'_x (\frac{L}{d})^2 \right] + \frac{1}{2} (C'_s + C_s - T'_x x - T_x) \\
\text{Mean } N_y &= k_1 k_3 u d \left[ 0.45 - 0.312 \varepsilon'_y (\frac{L}{d})^2 \right] + \frac{1}{2} (C'_s + C_s - T'_y y - T_y)
\end{align*}
\] (3.47, 3.48)
In (3.43) to (3.48), \( \varepsilon_x' \) and \( \varepsilon_y' \) are given by (3.19) and (3.20) and are the sum of the effects of axial strains and lateral boundary displacements at the ultimate load. The amount of variation of the actual membrane force per unit width from the mean value depends upon the particular case. For example, in a square unreinforced slab if \( \varepsilon_x' = \varepsilon_y' = 0 \) the maximum and minimum membrane forces per unit width differ by \( \pm 14\% \) from the mean value when all edges are fully restrained. Variations of this order are within the limits with which \( \varepsilon_x' \) and \( \varepsilon_y' \) can be estimated.

3.5 The Magnitude of the Axial Strains and Lateral Displacements.

To determine \( \varepsilon_x' \) and \( \varepsilon_y' \), given by (3.19) and (3.20), the magnitude of the axial strains, \( \varepsilon_x \) and \( \varepsilon_y \), and the lateral boundary displacements, \( t_x \) and \( t_y \), must be known. The lateral boundary displacements will depend upon the stiffness of the panels and beams surrounding the loaded slab and will be considered in detail in Chapter 5. The likely magnitudes of the axial strains, however, will be considered in this section.

\( \varepsilon_x \) and \( \varepsilon_y \) are the sums of the elastic, creep and shrinkage strains which have occurred at mid-depth of the
slab in the directions of the spans during the lifetime of the slab (which could be a few days or many years) and during the loading run to failure. The elastic strains may be found from the Young's modulus of the concrete (from short-term tests) and the mean membrane stress at mid-depth at ultimate load. This required membrane stress may be obtained by dividing the equations for mean membrane force per unit width, (3.45) to (3.48), by the slab depth. The magnitudes of the creep strains will depend upon the intensity of the working load sustained during the lifetime of the slab, and upon the extent of cracking of the concrete which occurs due to the sustained working load. If the sustained working load is a low proportion of the ultimate load there may be no cracking of the concrete during the period of sustained loading (due to the tensile bending stress not exceeding the ultimate tensile strength of the concrete), and under these conditions there will be no axial creep strains since the membrane forces within the slab will be zero. It is to be noted that membrane forces are only induced in the slab when the concrete cracks and the boundaries are forced to exert horizontal reactions acting inwards on the slab to balance the compressive stress in the concrete. If no cracking has occurred the compressive
stress in the concrete at each section of the slab is balanced by the tensile stress in the concrete at the section and since the vertical deflection of the slab is small at this stage the neutral axis will be at mid-depth. In practice most slabs will not be cracked at the working load. If the working load is high enough to cause cracks axial creep strains will occur. For the case of full cracking (i.e. cracking along all the yield lines) at a sustained working load the axial creep strains will be approximately proportional to the magnitude of the membrane stress (for the order of stresses involved) during the period of sustained loading, and it will be assumed that:

Mean membrane stress at mid-depth at a sustained working load \( w_s \) when the slab has cracked along the yield lines

\[
\frac{w_s}{w_u} x \frac{d}{2} x (\text{Mean } N_x \text{ at ultimate load}), \text{ for } x \text{ direction strips}
\]

or,

\[
\frac{w_s}{w_u} x \frac{d}{2} x (\text{Mean } N_y \text{ at ultimate load}), \text{ for } y \text{ direction strips}
\]

In \(3.49\), \( w_u \), \( \text{Mean } N_x \) and \( \text{Mean } N_y \) are values calculated for the stage when the sustained load is first applied. They may be obtained by substituting \( e'_x = e'_y = 0 \) into the equations
already developed for ultimate load and mean membrane forces since the effects of axial strains and boundary displacements when the sustained load is first applied are usually small. The shrinkage strains required for $\varepsilon_x'$ and $\varepsilon_y'$ will be the shrinkage which has occurred during the lifetime of the slab.

The determination of losses in prestressed concrete structures has required a knowledge of elastic, creep and shrinkage strains and the British code of practice for prestressed concrete lists values which could be used. The values are:

Elastic strain:

$$\varepsilon_e = \text{found using values for Young's modulus, } E_c, \text{ of } 3 \times 10^6, 4 \times 10^6 \text{ and } 4.5 \times 10^6 \text{ lb./sq.in. for concrete with cube strengths, } u, \text{ of } 3000, 4000 \text{ and } 5000 \text{ lb./sq.in., respectively.}$$

Creep strain:

$$\varepsilon_c = \text{creep strain per lb./sq.in. stress } = 0.25 \times 10^{-6} \times \frac{6000}{u}, \text{ when the stress is applied when the concrete is 2 to 3 weeks old and sustained indefinitely. For } u>6000 \text{ lb./sq.in. the value remains constant at } 0.25 \times 10^{-6}.$$

Shrinkage strain:

$$\varepsilon_s = 300 \times 10^{-6}.$$
The above strains will probably produce conservative design. For example the Comité Européen du Béton Recommendations\textsuperscript{13} quote for normal conditions rather higher values for Young's modulus (hence lower \(\varepsilon_e\) values), and rather lower values for \(\varepsilon_c\) and \(\varepsilon_s\) than the British code of practice. The creep and shrinkage strains in the C.E.B. recommendations, however, depend upon the relatively humidity, and with exceptionally dry conditions the recommended creep and shrinkage strains appreciably exceed the British recommendations. The British recommendations make no allowance for variation in humidity.

Using the values of the strains from the British code of practice and the membrane stresses from (3.45) to (3.49), the values of \(\varepsilon_x\) and \(\varepsilon_y\) at the ultimate load are (summing elastic, creep and shrinkage strains):

(a) For all edges fully restrained

\[
\varepsilon_x = \frac{1}{E_c} \left[ k_3 u \left( \frac{3}{8} + \frac{1}{16} \frac{L_x}{L_y} - \frac{\varepsilon_x'}{4} \frac{L_x^2}{d^2} \right) + \frac{1}{2d} (C' + C - T' - T_x) \right]
\]

\[
+ K \frac{w_s}{w_u} \text{with } \varepsilon_x' = \varepsilon_y = 0 \left[ k_3 u \left( \frac{3}{8} + \frac{1}{16} \frac{L_x}{L_y} \right) + \frac{1}{2d} (C' + C - T' - T_x) \right] \times \frac{1500 \times 10^{-6}}{u} + 300 \times 10^{-6}
\]

(3.50)
\[ \varepsilon_y = \frac{1}{E_c} \left\{ k_1 k_3 u \left( \frac{7}{16} - \frac{\varepsilon'_y}{4} \frac{L_y}{L_x} \right) + \frac{1}{2d} (C'_{sy} + C_{sy} - T'_{y} - T_{y}) \right\} + K \frac{w_s}{w_u} \text{with } \frac{\varepsilon'_x}{\varepsilon'_y} = 0 \left( k_1 k_3 u \frac{7}{16} + \frac{1}{2d} (C'_{sy} + C_{sy} - T'_{y} - T_{y}) \right) \times \frac{1500 \times 10^{-6}}{u} + 300 \times 10^{-6} \]  

(3.51)

Where \( K = 0 \) if the concrete is uncracked, or \( K = 1 \) if the concrete is fully cracked at the sustained load.

(b) For three edges fully restrained, one edge laterally unrestrained.

For the fully restrained spans:

\[ \varepsilon_x = \frac{1}{E_c} \left\{ k_1 k_3 u (0.4 + 0.05 \frac{L_x}{L_y} - 0.312 \varepsilon'_x \frac{L_x^2}{d^2}) + \frac{1}{2d} (C'_{sx} + C_{sx} - T'_{x} - T_{x}) \right\} + K \frac{w_s}{w_u} \text{with } \frac{\varepsilon'_x}{\varepsilon'_y} = 0 \left( k_1 k_3 u (0.4 + 0.05 \frac{L_x}{L_y}) \right) \times \frac{1500 \times 10^{-6}}{u} + 300 \times 10^{-6} \]  

(3.52)

\[ \varepsilon_y = \frac{1}{E_c} \left\{ k_1 k_3 u (0.45 - 0.312 \varepsilon'_y \frac{L^2}{d^2}) + \frac{1}{2d} (C'_{sy} + C_{sy} - T'_{y} - T_{y}) \right\} + K \frac{w_s}{w_u} \text{with } \frac{\varepsilon'_x}{\varepsilon'_y} = 0 \left( 0.45 k_1 k_3 u + \frac{1}{2d} (C'_{sy} + C_{sy} - T'_{y} - T_{y}) \right) \frac{1500 \times 10^{-6}}{u} + 300 \times 10^{-6} \]  

(3.53)
Where $K = 0$ if the concrete is uncracked, or $K = 1$ if the concrete is fully cracked at the sustained load.

The range of values of $\varepsilon_x$ and $\varepsilon_y$ for extreme cases of slabs with all edges fully restrained will now be examined.

(a) Maximum $\varepsilon_x$ and $\varepsilon_y$:

Consider a working load of $1/3$rd of the short-term ultimate load to be sustained by a slab indefinitely (in normal reinforced concrete design a load factor of 3 is used when the strength of the structure mainly dependent upon the compressive strength of the concrete). Let the slab be unreinforced and hence the membrane stresses will be the maximum possible. Let the full effect of shrinkage occur and the slab be fully cracked along the yield lines. From (3.50) it may be shown that $670 \times 10^{-6} < \varepsilon_x < 730 \times 10^{-6}$ for $\frac{L_x}{d} = 10$, and $540 \times 10^{-6} < \varepsilon_x < 580 \times 10^{-6}$ for $\frac{L_x}{d} = 40$, for ranges of span ratios and cube strengths of $1 < \frac{L_y}{L_x} < 2$ and $3000 < u < 5000$ lb/sq.in. From (3.51) it may be shown that $690 \times 10^{-6} < \varepsilon_y < 730 \times 10^{-6}$ for $\frac{L_y}{d} = 10$, and $470 \times 10^{-6} < \varepsilon_y < 580 \times 10^{-6}$ for $\frac{L_y}{d} = 40$, for the above ranges of span ratios and cube strengths.

(b) Minimum $\varepsilon_x$ and $\varepsilon_y$:

If the slab and boundary restraints undergo the same shrinkage and the slab is loaded directly to failure without
any period of sustained loading, the only contributions to $\varepsilon_x$ and $\varepsilon_y$ will be the elastic strains. Elastic strains are a minimum when membrane stresses are low, say a low cube strength of 3000 lb./sq.in. and a high tension steel content of 1% of mild steel placed top and bottom. Then it may be shown that $\varepsilon_x$ and $\varepsilon_y$ vary between $100 \times 10^{-6}$ and $190 \times 10^{-6}$ for $10 < \frac{d}{L} < 40, 1 < \frac{V}{L} < 2$ and $3000 < u < 5000$ lb./sq.in.

The above shows that $100 \times 10^{-6}$ to 730 $\times 10^{-6}$ might be considered to be the ranges of $\varepsilon_x$ and $\varepsilon_y$ to be expected in practice. Since the slab will be reinforced and since in most cases the cracking of the concrete due to the sustained working load will be small, the values of $\varepsilon_x$ and $\varepsilon_y$ generally found will be a great deal less than the maximum values calculated above.

3.6 **Reduction Coefficients to Allow for Axial Strains and Lateral Displacements.**

It is evident from section 3.5 that the strains $\varepsilon_x$ and $\varepsilon_y$ are difficult to determine accurately, and that the ultimate load equations involving axial strains are lengthy. A convenient method of obtaining the ultimate strength of slabs including the effect of axial strains and lateral boundary displacements is to find the ultimate load given by rigid-
plastic theory assuming rigid boundaries and then to multiply it by a reduction coefficient, \( R \), which takes the effect of \( \varepsilon_x' \) and \( \varepsilon_y' \) into account. Thus:

\[
W_u = R \times (w_u \text{ from theory with } \varepsilon_x' = \varepsilon_y' = 0) \quad (3.54)
\]

Where

\[
R = \frac{w_u \text{ from theory including } \varepsilon_x' \text{ and } \varepsilon_y'}{w_u \text{ from theory with } \varepsilon_x' = \varepsilon_y' = 0}. \quad (3.55)
\]

Values of the reduction coefficient \( R \) to cover the range of design cases can be calculated and plotted for easy reference. Figures 3.4 and 3.5 show values of \( R \) calculated from (3.55) for slabs with all edges fixed against rotation and with various \( L_y/L_x \) ratios, thicknesses and steel contents. In Fig.3.4 the cross-sectional area of reinforcement, per unit width, is the same in both directions. In Fig.3.5 the long span steel has the minimum cross-sectional area allowed by the British code of practice¹ and the short span steel is varied. It can be shown that minimum steel in the long span is nearly always the result when reinforcement is placed to give a design strength using the minimum weight of steel (see Appendix A). In the two figures \( \varepsilon_x' = \varepsilon_y' \) is considered since, especially for the steel distribution of Fig.3.4, there
FIG. 3.4 REDUCTION COEFFICIENTS FOR RECTANGULAR SLABS WITH ALL EDGES RESTRAINED AND WITH EQUAL STEEL TOP AND BOTTOM IN EACH DIRECTION.

NOTE: Reinforcement: $f' = 40,000$ lb./sq.in.

- $d'_{1x} = d'_{1y} = d'_{y} = 0.85d$

Compression steel neglected.

Concrete: $u = 3,000$ lb./sq.in.

From (3.55)
FIG. 3.5 REDUCTION COEFFICIENTS FOR RECTANGULAR SLABS WITH ALL EDGES RESTRAINED AND WITH 0.15% STEEL TOP AND BOTTOM IN DIRECTION OF LONG SPAN AND VARIOUS STEEL %'S IN DIRECTION OF SHORT SPAN.
will not be much difference between them, as is shown in the calculations at the end of section 3.5 for axial strains. Figures 3.4 and 3.5 show that for slabs with $L_x/d < 20$, $\varepsilon_x'$ and $\varepsilon_y'$ have negligible effect on the ultimate strength since $R$ is close to unity. If $\varepsilon_x'$ and $\varepsilon_y'$ are less than $400 \times 10^{-6}$, the effect is also negligible for slabs with $L_x/d < 30$. For slabs with $L_x/d = 40$ (the thinnest slabs, allowed by the British code of practice), $\varepsilon_x'$ and $\varepsilon_y'$ have considerable effect if these are greater than $200 \times 10^{-6}$. The kink in the curves for $L_x/d = 40$ is due to the governing equation, for ultimate flexural strength changing from (3.18) for small $\varepsilon'$ values to (3.23) for large $\varepsilon'$ values. As was explained in the theory leading up to (3.24), there really should be a smooth transition curve over a small range of $\varepsilon'$ between the two curves, but neglect of this is conservative. For slabs with $L_x/d < 30$, (3.18) governs the cases plotted in Figs. 3.4 and 3.5. The two figures also illustrate that large axial strains and fully restrained boundaries may cause the ultimate strength to be reduced to less than the Johansen's yield-line theory ultimate load in the case of thin slabs with high reinforcement contents. It is also of interest to note that for thick slabs reduction factors slightly greater than unity are possible due to the neutral axis shift caused by axial
strains and boundary displacements allowing a slightly more favourable combination of bending moment and axial force.

3.7 Experimental Work

3.7.1 Range of Experimental Results Available from Previous Investigations.

The tests on slabs with boundaries restrained against lateral displacement conducted in the past all fall into the category of short-term loading and hence creep and usually shrinkage have had little influence on the measured ultimate strengths. The slabs tested by Powell$^5$ and Wood$^6$ were all loaded to failure in a few hours. Ockleston$^{3,4}$ took two or three days to load each of his full scale interior panels up to ultimate strength. The University of Illinois 1/4-scale slab and beam floor$^7$ was tested in a large number of test runs but loading was left on the floor only long enough to record the gauge readings. Thus sustained loading tests over long periods of time with measurements of creep, shrinkage and boundary displacements have not been carried out in the past.

3.7.2. The Testing Arrangements and the Slabs

The details of the test frame (same as for the short-term load tests of slab series A and D) and the method of loading used are described in Appendix B. The slabs
were tested under upward pressure applied uniformly to the whole of the bottom face. Since the test frame was of mild steel it did not undergo creep or shrinkage during the period of sustained loading, and the elastic deformations of the test frame during the test runs were also small due to its extreme stiffness.

The details of the slabs are given in full in Appendix B and are summarized in Table 3.2. The slabs are referred to as series E. The spans of the slabs between supports were 60in. x 40in., and the nominal thickness was either 2in., 1.5in. or 1in. To develop the maximum membrane stresses within the slab, and hence to make the effect of elastic and creep strains a maximum, the slabs were unreinforced. For clamping purposes each slab was cast with a 12.5in. wide by 2in. thick concrete edge strip around the area to be tested. The slabs were tested with all edges restrained as much as possible. Rotation of the edge strips was prevented by channels pulled down on to the edge strips by steel studs. High strength cement grout was placed between the channels, edge strips and test frame to obtain even bearing. Horizontal movement of the edge strips was restrained by the bond due to the grout, by
friction induced by pulling the clamping channels down hard onto the edge strips, by bearing against the studs where they passed through the concrete, and by high tensile steel bolts bearing horizontally against steel plates at the edges of the slabs.

The slabs were loaded uniformly by a rubber bag filled with water at the required pressure. The sustained loading was applied by a static head of water provided by a water tank which was connected to the loading bag and raised to the required height above the loaded face of the slab.

3.7.3. Sequence of Casting and Loading

All slabs, except E1, were cured for 7 days after casting. Slab E1 was cured for 10 days. All slabs except E4 and E5, were clamped into the test frame when 10 days old. Slabs E4 and E5 were clamped into the test frame when 147 days old to minimize shrinkage. The grout which was used to obtain even bearing around the edge strips was given four days to harden before the slabs were loaded. The sequence of loading was:

1. The slabs were loaded up to the load to be sustained. (In the range 0.31 to 0.78 of the estimated short-term ultimate load). The time taken to apply this load was 15 to 50 minutes.
2. The sustained load was held constant for 42 days, except for slab El which failed under the sustained load in less than 1 day.

3. The loading on the slab was released and then increased immediately (except for slab El) until failure occurred, 15 to 55 minutes being taken to load each slab up from zero load to failure.

The vertical deflections of the slabs were measured at all stages of loading. Table 3.2 shows the details of the ages of the slabs at testing and the magnitudes of the sustained and ultimate loads. Figures 3.6 and 3.7 show the curves of load versus central deflection for the stages of loading. Figure 3.8 shows the rate of increase of deflection with time during the period of sustained loading.

A period of sustained loading of 42 days was chosen as being sufficient for creep and shrinkage to have an appreciable effect. In order to enhance the effect of creep the sustained loading applied was usually greater than practical working loads for the slabs.

3.7.4 Measurement of Elastic, Creep and Shrinkage Strains

Specimens were cast from the concrete mix used for the slabs to obtain representative values for Young’s modulus and the creep and shrinkage strains which occurred
Uniformly distributed loading.

lb./sq.in.

Under $w_s$ - 1st visible crack

Failure at $w_s$

SLAB E1 60" x 40" x 0.97"

Under $w_s$ - 1st visible crack

Release and reload

SLAB E2 60" x 40" x 0.95"

Under $w_s$ - 1st visible crack

Release and reload

SLAB E3 60" x 40" x 0.98"

SLAB E4 60" x 40" x 0.95"

SLAB E5 60" x 40" x 0.98"

FIG. 3.6 LOAD-DEFLECTION CURVES OF SLABS E1 TO E5.
FIG. 3.7  LOAD-DEFLECTION CURVES OF SLABS E6 TO E8.
FIG. 3.8 CENTRAL DEFLECTIONS OF SLABS OF SERIES E DURING PERIOD OF SUSTAINED LOADING.
during the test periods. The specimens were cured in the same manner as the slabs. Full details of the specimens, the methods of loading and measurement, and the recorded strains are given in Appendix B. The creep and shrinkage specimen were placed adjacent to the slab under load to ensure that the conditions of temperature and humidity were the same as for the slabs. Figure 3.9 shows a typical test set-up of slab under load and adjacent creep and shrinkage specimens. The values measured are summarized below. Since specimens were not cast with all slabs, mean values will be used in the calculations.

(a) Young's Modulus, $E_c$.

Longitudinal strains were measured on axially loaded cylinders to obtain $E_c$ at the stage of first applying the load to the slabs and at the stage of loading the slabs to failure. The readings were taken as quickly as possible to avoid creep. $E_c$ was determined from the tangent to the stress-strain curve at zero stress and the values calculated are given in Table B.6. Figure 3.10 shows the values obtained plotted against the cube strength of the concrete at the time of testing. The experimental points plotted in the figure show a good deal of scatter, indicating only a trend of increasing $E_c$ with increasing cube strength. Scatter of this nature has invariably been found by
FIG. 3.9 SLAB AND CREEP SPECIMENS UNDER SUSTAINED LOADING.
FIG. 3.10 VARIATION OF YOUNG'S MODULUS WITH CUBE STRENGTH OF CONCRETE.
investigators in the past, for example$^{18}$. Figure 3.10 also shows the recommendation for $E_c$ from CP 115$^{12}$ (discussed in section 3.5) plotted against cube strength. The CP 115 recommendations will be adopted for the calculations of the elastic strains in the slabs, since although the agreement with experiment is not good, the elastic strains will in any case be too small to have a significant effect on the total axial strains.

(b) Creep Strain, $\varepsilon_c$.

Longitudinal creep strains were measured on axially loaded cylinders under constant sustained stress at the age and for the period of time the slabs were under sustained loading. For example the specimens associated with slabs E2, E3, E6, E7 and E8 were loaded at age 14 days and left loaded for 42 days, and the specimens associated with slabs E4 and E5 were loaded at age 151 days and left loaded for 42 days. Shrinkage strains which occurred during the period of sustained loading were subtracted from the measured strains to give the true creep strains. The creep strain per lb./sq.in. stress (the specific creep strain) was calculated assuming that for the range of sustained stress applied, the creep strain was proportional to the applied stress. It is well known that this relationship is approximately true for
sustained stresses of up to approximately one half of the ultimate strength of the specimen. The creep strains per lb./sq.in. stress measured are shown in Table B.7. The mean of these values were:

For slabs first loaded at age 14 days:

\[ \varepsilon_c = 0.61 \times 10^{-6} \text{ per lb./sq.in. stress.} \]

For slabs first loaded at age 151 days:

\[ \varepsilon_c = 0.20 \times 10^{-6} \text{ per lb./sq.in. stress.} \]

In the slabs the concrete is subjected to a two-dimensional stress system and it may be thought that the use of creep strains found from tests on specimens under one-dimensional stress will lead to error. Ross\(^{15}\), however, has found that Poisson's ratio for creep appears to be zero, and hence that the creep strains found from one-dimension tests can be used to estimate the creep strains in the two directions of a two-dimensional stress system without consideration of interaction effects.
(c) Shrinkage Strain, $\varepsilon_s$.

Shrinkage strains were measured on either unloaded cylinders or small slab specimens. They were measured from the time of clamping the slab into the test frame until the end of the period of sustained loading. For the specimens associated with slabs E2, E3, E6, E7 and E8 this meant commencing readings at age 10 days and continuing readings for 46 days. For the specimens associated with slabs E4 and E5 readings were commenced at age 147 days and continued for 46 days. The measured shrinkage strains during these periods are shown in Table B.8. The mean values were:

For slabs clamped at age 10 days: $\varepsilon_s = 335 \times 10^{-6}$
For slabs clamped at age 147 days: $\varepsilon_s = 5 \times 10^{-6}$

3.7.5 Measurement of Lateral Movement of Slab Edges.

Complete restraint against lateral movement at the boundaries of the tested region of each slab was not obtained due to the finite stiffness of the test frame. Under the action of the membrane forces of the slab the sides of the test frame bowed outwards slightly. The stretch of the sides of the test frame due to direct ring tension, however, was negligible because of the large cross-sectional area of steel involved. Also, the horizontal forces from the slab
were transferred to the test frame through a finite width of the edge strips, and shortening of this width due to shrinkage and horizontal forces would tend to cause outward movement of the edges of the tested region. However, the width of the edge strip involved in the transfer of force from the slab to the test frame was probably small, and any tendency for the edge strips to decrease in width would be restrained by the clamping arrangements. These latter two possible causes of edge movement will therefore be ignored and only bowing of the sides of the test frame will be considered. Dial gauges measuring outward horizontal movement due to bowing were placed at the centre of each of the two long sides of the test frame, but not at the centre of the short sides. By an elastic analysis of the deformations of the test frame under the action of the reactions imposed by the slab and the loading bag it can be shown that the outward displacement of the centre of each long side was some six times that at the centre of each short side. This was due to the greater span of the long sides and also to the rotation of the ends of the long sides causing inward movement of the centres of the short sides which balanced a large proportion of the outward
movement of the short sides due to loading. In the theory developed for the slabs it was assumed that any outward movement of an edge, $t$, was constant along the edge. To include the effect of the bowing of the sides of the test frame it will be assumed that at each long side uniform outward movement of one half of the measured maximum movement (measured at the centre of the side) occurred. This assumed movement is approximately equivalent to the mean of the actual movement. The outward movement of the short sides of the test frame will be neglected since it was of such small magnitude. Table 3.1 shows the mean values of the outward displacement of each long edge, $t_x'$, as measured from each slab ($t_y'$ is assumed zero) when loaded from zero to maximum load.

3.8 Calculation of the Theoretical Ultimate Flexural Loads of the Test Slabs.

To calculate the theoretical ultimate loads of the slabs the values of the axial strains ($\varepsilon_x$ and $\varepsilon_y$) and of the lateral boundary displacements ($t_x$ and $t_y$) occurring during the period of loading are required in order that $\varepsilon'_x$ and $\varepsilon'_y$ (given by (3.19) and (3.20)) may be determined. The values for $t_x$ and $t_y$ have already been tabulated in Table 3.1.
\( \varepsilon_x \) and \( \varepsilon_y \) will be estimated using the following approximate equations based on section 3.5:

\[
\varepsilon_x = \left( \frac{1}{E_c} \right) x \text{Mean } N_x + K \left( \varepsilon_c \frac{w_s}{w_u} \right) x \text{Mean } N_x + \varepsilon_s
\]

(3.56)

\[
\varepsilon_y = \left( \frac{1}{E_c} \right) x \text{Mean } N_y + K \left( \varepsilon_c \frac{w_s}{w_u} \right) x \text{Mean } N_y + \varepsilon_s
\]

(3.57)

Where in (3.56) and (3.57), \( K = 0 \) if it is assumed that the concrete was uncracked when the sustained load was applied and hence no axial creep strains occurred, or \( K = 1 \) if it is assumed that the concrete was fully cracked along the yield lines and hence the maximum possible axial creep strains occurred. Also in (3.56) and (3.57), \( E_c, \varepsilon_c \) and \( \varepsilon_s \) are the values given in section 3.7.4 and the mean membrane forces are given by (3.45) and (3.46). The values of \( w_u \) and mean membrane force required for the determination of the creep strains may be found assuming \( \varepsilon'_x = \varepsilon'_y = 0 \).

Table 3.1 shows the values of \( \varepsilon'_x \) and \( \varepsilon'_y \) calculated from the values of \( \varepsilon_x \) and \( \varepsilon_y \) determined as above and from the tabulated values of \( t_x \) and \( t_y \). The values of \( \varepsilon'_x \) and \( \varepsilon'_y \) tabulated have been calculated for the extreme cases of no
axial creep strain (i.e. an uncracked slab) and maximum axial creep strain (i.e. a fully cracked slab). The extent to which cracking actually occurred in the slabs will be discussed later. On the basis of the estimated $\varepsilon'_x$ and $\varepsilon'_y$ values the theoretical ultimate loads of the slabs calculated from either (3.18) or (3.23) are shown in Table 3.2. It was only necessary to use (3.23) in the case of Slab E2 with $\varepsilon'_x$ and $\varepsilon'_y$ including creep. In all other cases (3.18) was the governing equation. The theoretical ultimate loads of the slabs assuming $\varepsilon'_x = \varepsilon'_y = 0$ at the stage of first load and when loaded to failure are also shown in Table 3.2 for comparison.

In all the theoretical ultimate load calculations the values of $k_1, k_3$ and $k_2$ used were those reported by Hognestad, Hanson and McHendry modified for cube strengths as shown in Fig.2.2. It should be noted that these coefficients were found from specimens tested under relatively short-term loading. Rusch has shown that high sustained loads on concrete can reduce the cube strength by as much as 20% and also alter the shape of the compressive stress block. Hence sustained loading could make significant difference to the $k_1, k_3$ and $k_2$ values. In the case of the slabs tested, however, the sustained load was not a high proportion of the
ultimate load (except for E1) and hence the short-term values for $k_1$, $k_3$, and $k_2$ should give sufficient accuracy.

Table 3.1: Edge Displacements and Axial Strains.
Slab Series E. At Ultimate Load.

<table>
<thead>
<tr>
<th>Slab Mark</th>
<th>Lateral Edge Displacement $\times 10^{-4}$ in.</th>
<th>$\varepsilon' = (\varepsilon + \frac{2t}{L}) \times 10^{-3}$ *</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_x$</td>
<td>$t_y$</td>
</tr>
<tr>
<td>E2</td>
<td>30</td>
<td>-</td>
</tr>
<tr>
<td>E3</td>
<td>33</td>
<td>-</td>
</tr>
<tr>
<td>E4</td>
<td>28</td>
<td>-</td>
</tr>
<tr>
<td>E5</td>
<td>23</td>
<td>-</td>
</tr>
<tr>
<td>E6</td>
<td>108</td>
<td>-</td>
</tr>
<tr>
<td>E7</td>
<td>103</td>
<td>-</td>
</tr>
<tr>
<td>E8</td>
<td>153</td>
<td>-</td>
</tr>
</tbody>
</table>

* From (3.19) and (3.20), with $\varepsilon_x$ and $\varepsilon_y$ from (3.56) and (3.57) with either $K = 1$ or $K = 0$.

Note: Strains were not measured for slab E1.
Table 3.2
<table>
<thead>
<tr>
<th>Slab Mark</th>
<th>$a$ in.</th>
<th>Age of slab when first loaded, days.</th>
<th>$u$ lb./sq.in.</th>
<th>(a) Sustained load. $w_a$ lb./sq.in.</th>
<th>(b) Load at failure $w_f$ lb./sq.in.</th>
<th>(c) Theoretical Ultimate Loads from (3.18) or (3.23) $w_{u \text{ theory}}$ lb./sq.in.</th>
<th>(d) $w_{u \text{ test}}$ at first load $w_u$ lb./sq.in.</th>
<th>(e) $w_{u \text{ test}}$ at failure $w_{u \text{ test}}$ lb./sq.in.</th>
<th>(f) $w_{u \text{ theory}}$ with creep</th>
<th>(g) $w_{u \text{ theory}}$ without creep</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>0.97</td>
<td>14</td>
<td>4790</td>
<td>4790</td>
<td>3.10</td>
<td>3.10</td>
<td>3.98</td>
<td>3.98</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>E2</td>
<td>0.95</td>
<td>14</td>
<td>3620</td>
<td>44.10</td>
<td>1.70</td>
<td>2.42</td>
<td>3.00</td>
<td>3.56</td>
<td>0.50</td>
<td>2.34</td>
</tr>
<tr>
<td>E3</td>
<td>0.98</td>
<td>14</td>
<td>3720</td>
<td>44.70</td>
<td>1.26</td>
<td>3.32</td>
<td>3.30</td>
<td>4.05</td>
<td>1.62</td>
<td>2.75</td>
</tr>
<tr>
<td>E4</td>
<td>0.95</td>
<td>151</td>
<td>5120</td>
<td>55.00</td>
<td>2.22</td>
<td>2.92</td>
<td>4.02</td>
<td>4.35</td>
<td>3.60</td>
<td>4.02</td>
</tr>
<tr>
<td>E5</td>
<td>0.98</td>
<td>151</td>
<td>6340</td>
<td>61.50</td>
<td>1.75</td>
<td>4.27</td>
<td>5.06</td>
<td>5.21</td>
<td>4.57</td>
<td>4.86</td>
</tr>
<tr>
<td>E6</td>
<td>1.48</td>
<td>14</td>
<td>4810</td>
<td>61.60</td>
<td>4.78</td>
<td>13.8</td>
<td>9.31</td>
<td>11.13</td>
<td>8.17</td>
<td>9.93</td>
</tr>
<tr>
<td>E7</td>
<td>1.54</td>
<td>14</td>
<td>4990</td>
<td>56.40</td>
<td>3.24</td>
<td>13.1</td>
<td>10.4</td>
<td>11.5</td>
<td>9.56</td>
<td>10.5</td>
</tr>
<tr>
<td>E8</td>
<td>2.07</td>
<td>14</td>
<td>4260</td>
<td>51.40</td>
<td>6.22</td>
<td>28.8</td>
<td>16.3</td>
<td>19.1</td>
<td>18.2</td>
<td>18.8</td>
</tr>
</tbody>
</table>

Using values of $c_x$ and $c_y$ from Table 3.1.

For all slabs: $I_x = 60$ in., $I_y = 40$ in.

Sustained loading was applied to all slabs for 42 days, except slab E1 which failed after less than 24 hours of loading.
3.9 Discussion

Columns (e) and (f) of Table 3.2 show the theoretical ultimate flexural loads of the slabs calculated including the effect of lateral boundary displacements and axial strains, with either full possible creep strains or zero creep strains. It is evident that axial creep strains will only occur if the concrete has cracked since only then will membrane stresses be induced to balance the compressive force in the concrete at the cracked sections. Thus column (e) will apply if the concrete had fully cracked along the yield line positions at the sustained load and column (f) will apply if the concrete was uncracked at that load. Figures 3.6 and 3.7 indicate the stage at which cracking became visible on the unloaded faces of the slabs. For slabs E2 to E8 there was very little visible cracking during the periods of sustained loading. Unfortunately the loaded face of each slab was hidden by the loading bag and hence the full extent of cracking could not be observed. For slabs E2 to E8, however, the lateral bowing of the sides of the test frame during the period of sustained loading was extremely small, if not zero, indicating that the membrane stresses and hence the creep strains were extremely small, if not zero. Thus for these slabs the theoretical ultimate loads should lie close to the values in column (f)
of Table 3.2. The reduction in ultimate load due to axial strains and lateral boundary displacements postulated by the theory can be seen by comparing columns (f) and (d). It is evident that for thin slabs \( \frac{L}{d} = 40 \) the reduction can be significant, but for thick slabs \( \frac{L}{d} = 20 \) the reduction is negligible.

In the case of slab E1 the sustained load was a high proportion (78%) of the theoretical ultimate load at the stage of first loading, and at the sustained load cracks had fully developed along the yield lines. Slab E1 failed after less than 24 hours under the sustained load. This was not surprising since large membrane forces would have been operative over the whole of the slab, due to the high load and extensive cracking, and hence the creep strains would have become large in magnitude very rapidly. The sustained loads applied to the other slabs varied between 56% and 31% of the theoretical short-term ultimate loads at the time. The sustained loads were therefore generally greater than practical working loads. Since for the slabs with the smallest sustained loads (E3, E5 and E7) there was no visible cracking of concrete or lateral bowing of the sides of the test frame during the periods of sustained load it is felt that in practice working loads on slabs will not cause axial creep strains.
Comparison of the theoretical ultimate loads of slabs E2 to E8 (column (f) of Table 3:2) with the ultimate loads measured in the tests shows that the theory is conservative for five out of seven results. The result for slab E8 is remarkable in that the observed ultimate load was 1.53 times the theoretical ultimate load calculated as above, and even 1.51 times the theoretical rigid-plastic ultimate. A discrepancy of this order between test and theory did not occur in the case of the slabs with dimensions similar to E8 tested under short-term loading (Chapter 2). The result for E8 can be partly explained in that there was some scatter of the cube strengths of the specimens cast with the slab. The mean cube strength was used in the calculation and the cube strength of the concrete from the top of the slab was 15% below the mean. Another explanation of the high test load of slab E8 (and some of the other slabs) could be that the creep strains occurring in the compressed concrete near the face of the slab (not to be confused with the axial creep strains discussed above) during the period of sustained loading allowed a more complete redistribution of bending moments to take place along the yield lines at ultimate load. The theory assumes that the concrete is
sufficiently plastic in compression to allow the concrete at the first yield lines to retain its full compressive strength while the remainder of the yield lines are developing. This assumption cannot be completely justified, however, and there will be some drop in stress in the concrete at the early yield lines before the last yield line develops. The assumption is compensated by assuming that the concrete develops only the uniaxial ultimate strength at yield lines rather than the greater biaxial value which could develop in a slab. Now creep of compressed concrete tends to reduce maximum stresses and thus when a slab is loaded to failure after a period of sustained loading the difference in load level between the formation of the first and the last yield lines will not be so great, and thus the concrete will show more tendency to reach its full biaxial strength. Only in the case of slabs E4 and E5 were the ultimate loads found from the tests less than the theoretical ultimates. This seems to indicate that the axial strains assumed for these two slabs in the calculations were under-estimated. The sustained load was applied when the two slabs were 151 days old and the associated specimens indicated shrinkage strains of only $5 \times 10^{-6}$ during the period of 46 days the slabs were clamped in the test frame.
In view of the assumptions that have been made in the theory in order to obtain a workable solution, and considering that the strength of the slabs was entirely dependent upon the compressive strength of the concrete (a quantity which invariably shows a fair amount of scatter, to the extent that the code assumes a strength of only 2/3rds of the supposed cube strength in ultimate load calculations), it is felt that the agreement between test and theory are reasonable.

In developing the theoretical ultimate load equations it was assumed that the slabs reach their ultimate flexural strength at central deflections of 0.5 of the slab thickness. The accuracy of this assumption for slabs loaded to failure after a period of sustained loading may be checked by referring to the load-deflection curves of Figs. 3.6 and 3.7. These curves show that in cases where the increase in deflection during the period of sustained loading was small the slabs reached ultimate load at approximately the anticipated deflection. Where the increase in deflection at the sustained load was large the slabs reached ultimate load at central deflections which were slightly greater than 0.5 of the slab thickness, but it is evident that this did not cause much reduction in the ultimate strength since the load-deflection curves show that slabs (especially thin ones)
continue to carry loads near maximum at deflections greater than 0.5 of the slab thickness. It is to be noted that since practical working loads are less than the sustained loads applied to the slabs, the central deflection at the ultimate load in practice should approach the assumed value.

Figure 3.11 shows the yield-line pattern of slab E8 at the end of the test. The other slabs exhibited similar yield-line patterns and it is evident that the assumption of 45° corner yield lines made in the theory is reasonable.
FIG. 3.11  SLAB E8 AT END OF TEST.
CHAPTER 4

THE TENSILE MEMBRANE BEHAVIOUR OF UNIFORMLY LOADED RECTANGULAR REINFORCED CONCRETE SLABS WITH ALL OR SOME EDGES LATERALLY RESTRAINED

4.1 Scope of Theory and Tests

Theory will be developed to define the ascending portion of the load-deflection curve at large deflections (the curve CD of Fig. 1.1) in order that the load-carrying capacity of the slab as a tensile membrane may be obtained. Slabs with either all edges fully fixed against translation or with three edges fully fixed against translation and the remaining edge free to translate horizontally will be considered. These two cases of boundary conditions may be regarded as the ideal cases of interior panels and edge panels, respectively, of a continuous slab and beam floor.

The theory developed will be checked against the test results obtained by Powell, and against the results from slab series A, B and C.

4.2 Stage of Behaviour

After the ultimate flexural strength of a reinforced concrete slab with fully restrained edges has been reached
the load carried by the slab decreases rapidly with further deflection due to the compressive membrane forces becoming smaller. At large deflections a stage is eventually reached where the membrane forces in the central region of the slab change from compression to tension and the slab boundary restraints commence to resist inward movement of the edges. At this stage, due to the large stretch of the slab surface, the cracks in the central region penetrate the whole thickness of the concrete and yielding of the reinforcement spreads throughout the region. With further deflection the load carried by the slab commences to increase since the large deflection enables the tensile membrane action of the reinforcement to support considerable load. The load carrying capacity of the slab continues to increase with further deflection and only becomes exhausted when the reinforcement commences to fracture.

In the case of a slab which is fixed against translation around all edges and which contains different amounts of steel in the x and y directions, the reinforcement will tend to act as a tensile membrane with different forces per unit width in the directions of the spans. In the case of a slab which has one edge which is free to translate horizontally, the remaining edges being fixed against
translational, the tensile membrane forces which develop in the direction normal to the laterally unrestrained edge will be small, and such a slab will therefore develop full membrane action in only one direction. Since in general the tensile membrane forces in the \( x \) and \( y \) directions in both cases will be different, it is to be noted that Prandtl's simple membrane equations are not applicable.

### 4.3 Theory For Load Carried by Tensile Membrane Action

#### 4.3.1 The Assumptions

In the analyses to follow it will be assumed that at the stage when the load carrying capacity of the slab as a tensile membrane becomes important:

(a) All the concrete has cracked throughout its depth and hence is incapable of carrying any load.

(b) All the reinforcement which spans the full distance between the fixed boundary has reached yield stress and carries the load as a plastic membrane.

(c) No strain hardening of steel occurs.

The first assumption is along the lines of conventional reinforced concrete theory which neglects the strength of concrete in tension. There will always be some uncracked concrete around the edges of the slab, however,
although in the case of heavily reinforced slabs the amount of uncracked concrete will be very small. For lightly reinforced slabs it may be found that the tensile strength of the concrete slab section may be greater than the tensile resistance of the steel and the only cracks which form are those due to bending earlier in the loading range. The assumption that all the concrete has cracked throughout its depth, however, is conservative, and it is difficult to see how the uncracked concrete could be taken into account since its extent is indeterminate.

The accuracy of the second assumption regarding the steel having reached yield stress will now be examined. If, as a first approximation, the deflected shape of each reinforcement bar is taken to be a parabola and \( f \) is the central sag of the bar, \( L \) is the horizontal distance between supports and \( S \) is the curved length of the bar, it can be shown that

\[
S = L \left[ 1 + \frac{8}{3} \left( \frac{f}{L} \right)^2 \right]
\]

\[
\therefore \text{Extension of the bar is}
\]

\[
S - L = \frac{8}{3} L \left( \frac{f}{L} \right)^2.
\]
Now since the ratio $f/L$ is small, the variation in tension along the length of the bar will also be small, and hence yielding will commence at all points along the bar at approximately the same load. Thus if $f'_s$ and $E_s$ are the yield stress and Young's modulus for the bar material respectively, yielding will have just commenced when:

$$\frac{f'_s}{E_s} = \frac{S-L}{L} = \frac{8(f'_s)^2}{3L^2},$$

or when

$$\frac{f}{L} = 0.612 \sqrt{\frac{f'_s}{E_s}}.$$

If for mild steel reinforcement typical values of $f'_s = 40,000$ lb./sq.in. and $E_s = 30 \times 10^6$ lb./sq.in. are taken, then

$$\frac{f}{L} = 0.612 \sqrt{\frac{40,000}{30 \times 10^6}} = \frac{1}{45}.$$  

It is to be noted that this value of the central sag to span ratio at the onset of yielding can only be regarded as an approximation, since before the tensile membrane action commences the steel has already yielded across the yield lines and so the extension of the bar is not entirely due to uniform stretching along its length. However, as
the short span of slabs will always be at least 40 times the thickness of the slabs (a requirement of the Code of Practice\textsuperscript{1} to ensure adequate stiffness at working load), and since at the stage at which the load carried by the tensile membrane becomes important the central deflection of the slab is at least twice the slab thickness, it is evident that most of the steel in the direction of the short span will be yielding (the exception being that steel which runs close to the edge). If the long span is much greater than the short span, the steel in the direction of the long span may not yield until the central deflection becomes greater, but this will not cause excessive overestimation of the load carried by a two-way membrane, since for such a slab the amount of long span steel will be small and its contribution to the load carried will be negligible whether it is yielding or not. It is to be noted that the second assumption also neglects the contribution to the strength of the slab of reinforcement which does not cross the whole span, for instance the top steel placed around the edges of slabs to carry the hogging bending moments in the flexural stage. The neglect of this steel is conservative.

The third assumption, the neglect of strain hardening of steel, is also conservative and is common to most plastic theories.
4.3.2 **Uniformly Loaded Two-Way Rectangular Slab Fixed Against Translation at All Edges.**

The reinforcement net is without resistance to shear and it will be analysed as a plastic membrane supported from a rigid rectangular boundary which lies in the \( x, y \) coordinate plane. (See Fig. 4.1). The yield forces, per unit width, of \( x \) and \( y \) direction steel which cover the whole area of the slab are \( F_x \) and \( F_y \), respectively. In general \( F_x \neq F_y \). Let the membrane be loaded by a uniform pressure \( w \) per unit area and let the vertical deflection \( z \) be positive if it is in the direction of the load. Consider a small rectangular element to be cut from the membrane by two pairs of planes parallel to the \( xz \) and \( yz \) coordinate planes. For the element, equilibrium of forces in the \( z \) direction (see Fig. 4.1) requires:

\[
0 = w \ dx \ dy - F_x \ dy \ \frac{\partial z}{\partial x} + F_x \ dy \left( \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} \ dx \right) \\
- F_y \ dx \ \frac{\partial z}{\partial y} + F_y \ dx \left( \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} \ dy \right)
\]

From which:

\[
\frac{\partial^2 z}{\partial x^2} + \frac{F_y}{F_x} \frac{\partial^2 z}{\partial y^2} = - \frac{w}{F_x}.
\]
Plastic Tensile Membrane

FIG. 4.1 UNIFORMLY LOADED PLASTIC MEMBRANE

Forces Acting On Element

FIG. 4.2 EQUIVALENT SIMPLE MEMBRANE
It is to be noted that the loading \( w \) considered above acts normal to the slab surface, but for the order of deflections considered the difference between vertical loading and normal loading is negligible. The relationship between load and deflection for a given reinforcement mesh is given by (4.1). A value of \( z \) must be found which satisfies (4.1) and gives \( z = 0 \) at the boundaries. To solve (4.1) replace the independent variable \( y \) by another independent variable \( Y \) such that:

\[
Y = y \sqrt{\frac{F_x}{F_y}}. \tag{4.2}
\]

Then

\[
\frac{\partial Y}{\partial y} = \sqrt{\frac{F_x}{F_y}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial y} = \sqrt{\frac{F_x}{F_y}} \frac{\partial z}{\partial Y}.
\]

Also

\[
\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \sqrt{\frac{F_x}{F_y}} \frac{\partial z}{\partial Y} \right) = \frac{\partial}{\partial Y} \frac{\partial Y}{\partial y} \left( \sqrt{\frac{F_x}{F_y}} \frac{\partial z}{\partial Y} \right) = \frac{F_x}{F_y} \frac{\partial^2 z}{\partial Y^2}.
\]

Hence

\[
\frac{F_y}{F_x} \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial Y^2}. \tag{4.3}
\]
On substituting (4.3) into (4.1),

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{w}{Fx} \cdot$$  \hspace{1cm} (4.4)

Now (4.4) is the differential equation for an equivalent simple membrane which has the same uniform force, per unit width, $F_x$ in each direction and has the dimensions and coordinates shown in Fig. 4.2. It can be seen that the half-lengths of the sides of the equivalent simple membrane are given by:

$$a = \frac{Lx}{2} \cdot$$  \hspace{1cm} (4.5)

And from (4.2)

$$b = \frac{Ly}{2} \sqrt{\frac{Fx}{Fy}} \cdot$$  \hspace{1cm} (4.6)

Timoshenko\textsuperscript{17} has shown that a value of $z$ which satisfies (4.4) and which satisfies the requirement of $z = 0$ at the boundaries is given by the following series:

$$z = \frac{16}{Fx \pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}(-1)^{\frac{n-1}{2}} \left(1 - \frac{\cosh \frac{nty}{2a}}{\cosh \frac{ntb}{2a}}\right) \cos \frac{ntx}{2a} \quad (4.7)$$
To obtain the deflection of the actual membrane substitute into (4.7) the values of $Y$, $a$ and $b$, given by (4.2), (4.5) and (4.6), respectively. Then:

$$z = \frac{4wL^2}{F_x \pi^2 \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} (-1)^{n-1} \left(1 - \frac{\cosh \left[ \frac{n \pi Y \sqrt{F_x}}{F_y} \right]}{\cosh \left[ \frac{n \pi L_x \sqrt{F_x}}{F_y} \right]} \right) \cos \frac{n \pi x}{L_x}$$

(4.8)

The central deflection, $\Delta$, of the actual membrane is given when $x = y = 0$.

Putting $x = y = 0$ into (4.8) and rearranging the terms, the relationship between $w$ and $\Delta$ follows:

$$\frac{wL^2}{F_x \Delta} = \frac{\pi^2}{4 \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} (-1)^{n-1} \left(1 - \frac{1}{\frac{n \pi L_x \sqrt{F_x}}{F_y}} \right) \cosh \left[ \frac{n \pi x \sqrt{F_x}}{F_y} \right]}$$

(4.9)

(4.9) gives a linear relationship between $w$ and $\Delta$ and defines the portion of the load-deflection curve between points C and D of Fig.1.1. The limiting point D will depend upon the ductility of the steel. The summation of the series in (4.9) can involve heavy computation and in
FIG. 4.3  LOAD-DEFLECTION RELATIONSHIPS FOR UNIFORMLY LOADED RECTANGULAR PLASTIC TENSILE MEMBRANES. FROM (4.9)
order to simplify the use of this equation values of \( \frac{wL^2}{\Delta F_x} \) obtained from the right hand side of it have been plotted in Fig. 4.3 for various \( \frac{L_y}{L_x} \) and \( \frac{F_x}{F_y} \) ratios. Only \( F_x > F_y \) for slabs with \( L_y > L_x \) have been considered since economy of steel will always require more steel in the direction of the short span than in the direction of the long span.

The curves illustrate that as the \( \frac{L_y}{L_x} \) ratio increases the load carried by the long span reinforcement quickly becomes negligible and the membrane tends to the "one-way" membrane case.

4.3.3. Uniformly Loaded Two-Way Rectangular Slab Fixed Against Translation at Three Edges and Free to Translate Horizontally at the Remaining Edge.

The load carried by the reinforcement which runs normal to the laterally unrestrained edge will be small due to the horizontal translation and doubtful anchorage conditions in the concrete at that edge. It will be assumed that the load is carried only by the steel which runs at right angles to the pair of parallel fixed edges. The central deflection, \( \Delta \), for this "one-way" membrane is given by the well known equation for a uniformly loaded cable:
If \( x \) direction steel carries all the load:

\[
\Delta = \frac{wL^2}{\delta F_x}.
\] (4.10)

If \( y \) direction steel carries all the load:

\[
\Delta = \frac{wL^2}{\delta F_y}.
\] (4.11)

4.4 Experimental Work

4.4.1 Range of Experimental Results Available from Past Investigations.

Powell\(^5\) and Wood\(^6\) have reported results for slabs acting as tensile membranes with all edges fully restrained. Wood's results, however, are not reported in enough detail to compare theory and experiment and no attempt will be made to analyse them. Powell's experimental load-deflection curves will be analysed and also those obtained from the slabs of series A, B and C.

4.4.2 The Testing Arrangements and Slabs

Details of the testing arrangements, method of loading and the slabs of series A, B and C are given in Appendix B. Details of the slabs of series A are summarized in Table 4.1.
### TABLE 4.1
Tests and Theory: Tensile Membrane Behaviour of Slabs With Fully Restrained Edges.

<table>
<thead>
<tr>
<th>Authority</th>
<th>Slab Mark</th>
<th>% of Reinforcement*</th>
<th>F** x lb/in.</th>
<th>F** y lb/in.</th>
<th>w from(4.9) at Δ=0.1L x lb/sq.in.</th>
<th>w from test at Δ=0.1L x lb/sq.in.</th>
<th>w test/wtheory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Powell</td>
<td>S47</td>
<td>0.39</td>
<td>152</td>
<td>152</td>
<td>6.82</td>
<td>9.0</td>
<td>1.32</td>
</tr>
<tr>
<td></td>
<td>S50</td>
<td>0.70</td>
<td>276</td>
<td>276</td>
<td>12.3</td>
<td>17.3</td>
<td>1.41</td>
</tr>
<tr>
<td></td>
<td>S54</td>
<td>1.11</td>
<td>436</td>
<td>436</td>
<td>19.5</td>
<td>25.9</td>
<td>1.33</td>
</tr>
<tr>
<td></td>
<td>S58</td>
<td>1.52</td>
<td>720</td>
<td>720</td>
<td>32.2</td>
<td>43.6</td>
<td>1.35</td>
</tr>
<tr>
<td></td>
<td>S59</td>
<td>1.52</td>
<td>720</td>
<td>720</td>
<td>32.2</td>
<td>43.6</td>
<td>1.35</td>
</tr>
<tr>
<td></td>
<td>S62</td>
<td>2.39</td>
<td>1136</td>
<td>1136</td>
<td>50.8</td>
<td>61.4</td>
<td>1.21</td>
</tr>
<tr>
<td></td>
<td>S63</td>
<td>2.39</td>
<td>1136</td>
<td>1136</td>
<td>50.8</td>
<td>65.6</td>
<td>1.29</td>
</tr>
<tr>
<td>Park</td>
<td>A1</td>
<td>0.16</td>
<td>163</td>
<td>163</td>
<td>4.04</td>
<td>9.2</td>
<td>2.28</td>
</tr>
<tr>
<td></td>
<td>A2</td>
<td>0.35</td>
<td>280</td>
<td>153</td>
<td>6.12</td>
<td>15.2</td>
<td>2.48</td>
</tr>
<tr>
<td></td>
<td>A3</td>
<td>0.59</td>
<td>603</td>
<td>163</td>
<td>12.3</td>
<td>20.8</td>
<td>1.69</td>
</tr>
<tr>
<td></td>
<td>A4</td>
<td>0.96</td>
<td>870</td>
<td>163</td>
<td>17.5</td>
<td>25.6</td>
<td>1.46</td>
</tr>
</tbody>
</table>

* Total cross-sectional area of steel (top plus bottom) which runs across the whole span as a % of the gross cross-sectional area of concrete.

** Based on the above % of reinforcement and yield stresses of the steel of either 30,600 or 37,000 lb./sq.in. for Powell's slabs, or in the range 45,000 to 50,000 lb./sq.in. for the slabs of series A.

Slab dimensions: Powell: 36in. x 20.57in. x 1.286in.
Series A: 60in. x 40in. x 2in.

Cube strengths of concrete: In range 5020 to 8130 lb./sq.in.
4.5 Discussion

4.5.1 Slabs With All Edges Fully Restrained

The theoretical (from (4.9)) and experimental load-deflection curves obtained from the slabs of series A are shown in Fig. 2.17. Figure 4.4 shows the extent of cracking on the unloaded faces of the slabs of series A. Table 4.1 sets out the theoretical and experimental results. The theory is seen to be conservative. In the cases of the lightly reinforced slabs it is evident that pure membrane action did not occur since, as Fig. 4.4 shows, the cracking present at the ends of the test runs of slabs A1 and A2 was little more than the cracking which developed with the yield-line pattern at the ultimate flexural load. Hence for the lightly reinforced slabs the load was carried by a stronger combined bending and tensile membrane action. The heavily reinforced slabs, however, cracked over a good portion of their area, as is shown by slabs A3 and A4 in Fig. 4.4, and therefore approached pure tensile membrane action. The assumption of no strain hardening also causes the theory to be conservative. The steel in Powell's slabs in particular showed considerable strain hardening soon after the yield stress was reached and the ultimate tensile strength of the reinforcement was
FIG. 4.4 UNLOADED FACES OF SLAB SERIES A AT END OF TEST.
1.6 to 1.8 times the yield stress. In the case of the steel in the slabs of series A this range was 1.2 to 1.4. Some of the steel in the slabs was near its ultimate tensile strength at the ends of the test runs, since fracture of the bars occurred in some cases, and thus a considerable increase in strength due to strain hardening was possible. Also, in the slabs of series A, top steel had been placed extending into only the end portions of the spans (to provide for the edge bending moments developed during the flexural stage), and this steel was not included in $F_x$ and $F_y$ in the calculations since it was not placed over the whole of the slab. It is apparent that this top steel (which was double the cross-sectional area of the bottom steel) strengthened the slabs considerably around the edges and was partly responsible for the very conservative theoretical results of the slabs of series A. In Powell's slabs both top and bottom steel were included in the strength calculations since the top steel was carried over the whole area of the slabs.

Central deflections of 1.9 to 2.4 times the slab thickness occurred in slabs A1 and A2, respectively, before fracture of the reinforcing bars commenced. In the case of slabs A3 and A4 the reinforcement had not fractured at central deflections of 3.0 and 2.3 times the slab thickness,
respectively, when loading was stopped due to danger of splitting the loading bag across the wide cracks. The $L_x/d$ ratio of these slabs was 20, and hence a safe value for the central deflection at the end of tensile membrane action could be taken to be $\frac{1}{10}$ of the short span. The stage at which fracture occurred in Powell's slabs is unknown, but Powell's experimental load-deflection curves were drawn up to central deflections of at least $\frac{1}{8}$ of the short span. It is of interest to compare the theoretical tensile membrane load at the suggested maximum deflection with the theoretical ultimate flexural load. Consider a square slab with fully restrained edges which is reinforced with mild steel with a yield stress of 40,000 lb./sq.in. placed equally top and bottom over the whole area of the slab in both directions. Let the distances between the centroids of top and bottom steel be 0.6 of the slab thickness and the cube strength of the concrete be 3000 lb/sq.in. For the short-term ultimate flexural load from (2.78) to be equal to the tensile membrane load from (4.9) at a central deflection of $\frac{1}{10}$ of the span, the total reinforcement content (top plus bottom) required in each direction for a span/depth ratio of 40 is 0.46% of the gross cross-sectional area of the concrete. For a span/
depth ratio of 20 this figure increases to 1.44%. These percentages will be somewhat less if the steel is more ductile and allows larger deflections of the slab before fracture. That heavily reinforced slabs have a reserve of strength over the ultimate flexural load was shown by Powell's slabs S58, S59, S62 and S63 which carried loads by tensile membrane action which exceeded the ultimate flexural loads. If the effect of sustained loading is to reduce the ultimate flexural load due to axial elastic, creep and shrinkage strains reducing the compressive membrane action, the ultimate tensile membrane load will compare more favourably with the ultimate flexural load since tensile membrane action is uneffected by long-term loading.

The theory also indicates that slabs with draped reinforcement (as is used by some American designers) will have considerable tensile membrane strength due to the large load carrying capacity of the reinforcement at relatively small deflections of the slab.

4.5.2 Slabs With Three Edges Fully Restrained and One Edge Free to Translate Horizontally

The theoretical (from (4.10) and (4.11)) and experimental load-deflection curves obtained from the slabs of series B and C are shown in Figs. 2.20 and 2.21.
Figures 4.5 and 4.6 show the extent of cracking on the unloaded faces of the slabs at the end of the test runs. Where experimental results are available the theory is shown to be conservative. The slabs of series C and slab B1 were not loaded into the tensile membrane stage due to the danger of splitting the loading bag across the wide cracks that developed along the sagging moment yield lines near the simply supported edge. Thus the crack patterns of these slabs are the yield-line patterns for the ultimate flexural load. Slabs B2, B3 and B4 were loaded into the tensile membrane stage, however, and considerable cracking developed as is evident from Fig. 4.5. The conservative nature of the theoretical curves is due to the reasons discussed above for the slabs with all edges fully restrained, and also to the assumption of one-way membrane action. The reinforcement running at right angles to the simply supported edge must contribute to the strength of the membrane in the regions of the slab away from that edge since the directions of the lines of cracking indicated two-way membrane action in those regions.
FIG. 4.5 UNLOADED FACES OF SLAB SERIES B AT END OF TEST.
FIG. 4.6 UNLOADED FACES OF SLAB SERIES C AT END OF TEST.
CHAPTER 5

THE LATERAL STIFFNESS AND STRENGTH REQUIRED TO ENFORCE MEMBRANE ACTION IN UNIFORMLY LOADED REINFORCED CONCRETE SLAB AND BEAM FLOORS WITH RECTANGULAR PANELS

5.1 Scope of Theory and Tests

In the previous chapters ultimate strength theory including membrane action was developed for uniformly loaded single panel rectangular concrete slabs which were restrained against movement at the boundaries. If membrane action is to be utilized in the design of panels of slab and beam floors the movement at the edges of each panel must be restricted by the stiffness and strength of the floor system. Ideally, the edge restraint should prevent rotation and vertical and horizontal translation. Of these, the restriction of horizontal translation is the most critical. It is well known that continuity between panels at supporting beams is sufficient to justify treatment of the edges as if fully fixed against rotation in plastic theory. Similarly, the ultimate strength of the panel is insensitive to small relative vertical translations at the edges caused by sagging of the supporting beams in the vertical plane. The lateral stiffness and strength of surrounding beams and panels has to be examined much more closely, however, since membrane action
is dependent upon the restriction of very small horizontal translations, and very large horizontal forces are involved. Very little has been reported on the lateral stiffness and strength of surrounding beams and panels. No doubt this has been due to the expense and labour of making up and testing to failure complicated slab and beam floor systems. Reports on loading tests of only two multi-panel floors are available. Ockleston\textsuperscript{3,4} has described tests on a full scale slab and beam floor, and the University of Illinois\textsuperscript{7} has reported on tests conducted on a $\frac{1}{4}$-scale model of a slab and beam floor. No theory was developed in these reports to account for the lateral stiffness and strength of the floor, but membrane action was observed.

This chapter considers the requirements for lateral stiffness and strength of slab and beam floors. The test results obtained by Ockleston and the University of Illinois will be examined. In addition, tests on small scale mortar models of idealized nine panel (arranged three by three) slab and beam floors will be described and the results analysed.
5.2 Prerequisites for Compressive Membrane Action in Slab and Beam Floors.

Figure 5.1 shows a typical reinforced concrete slab and beam floor consisting of panels cast monolithic with supporting beams. The beams lie on a rectangular grid (as is common in most floors), and thus the slab is divided into rectangular panels. The floor is normally supported by columns at the node points of the beams. Consider the floor to be loaded uniformly over the whole of its area until the ultimate flexural load is reached, and let the design of the floor be such that failure occurs over the whole of its area at the same load. The type of collapse mechanism developed will depend upon the relative flexural strengths of the beams and the panels. If the beams have a low ultimate strength a general collapse mechanism of the type shown in Fig. 5.2a will develop. In this collapse mechanism plastic hinges form in both the beams and the panels and the floor folds across its whole width into a series of "valleys". Membrane action cannot develop in this collapse mechanism since no restraint against lateral displacement can develop, and therefore the ultimate moments in the panels and the beams are the simple yield moments without membrane stresses. If the beams have a high ultimate strength a collapse mechanism of the type shown in Fig. 5.2b will develop.
FIG. 5.1 SLAB AND BEAM FLOOR.

- Plastic hinge in beam
- Sagging moment yield line in panel.
- Hogging moment yield line in panel.

FIG. 5.2 COLLAPSE MECHANISMS FOR UNIFORMLY LOADED SLAB AND BEAM FLOORS.
In this mechanism the failure is confined to the panels, the beams being strong enough to carry the ultimate loads of the panels without failing themselves. As far as analysis by plastic theory is concerned the panels may be considered to be supported by a rigid grid of beams. Membrane action can develop in the panels of this mechanism since horizontal restraint is provided by the grid of beams and the surrounding panels. Traditionally, the beams of slab and beam floors have been designed to support the ultimate loads of the panels and thus the collapse mechanism shown in Fig. 5.2b has been the one considered. It should be noted that the ultimate load of the floor given by the collapse mechanism of Fig. 5.2a should also be checked to ensure that it does not provide a lower load factor.

Consider the degree of restraint against horizontal displacement available to the panels of the collapse mechanism of Fig. 5.2b. For an interior panel of the type marked A it is evident that full membrane action could develop in the direction of each span due to the presence of beams and panels around all edges. For an exterior panel of the type marked B the lack of horizontal restraint at the outside edge makes the membrane forces acting normal to that edge small (the edge beam alone would be insufficiently stiff to apply full
lateral restraint), and membrane action could only be expected to develop in the direction parallel to the outside edge. In a corner panel of the type marked C the membrane action in both directions will be negligible since only edge beams are present at the two outside edges. Thus the interior panel A is capable of full membrane action, the exterior panel B is capable of one-way membrane action with simple bending in the other direction, and the corner panel C is incapable of any membrane action and will have simple bending in both directions. Figure 5.3 shows the horizontal forces due to this assumed membrane action at the ultimate flexural load. The forces act outwards on the supporting beams. Due to these forces the beams act as ties resisting the tendency for the segments of the panels to push outwards. Some beams will also bow laterally (in the horizontal plane) as shown in Fig.5.3. It is evident that if the assumed membrane action is to develop in the floor the panels and beams should have enough stiffness to restrain the extension and lateral bowing of the beams. The floor should also have sufficient strength to resist the large horizontal forces involved. These requirements for the assumed membrane action will now be considered in detail:
FIG. 5.3 COMPRESSIVE MEMBRANE FORCES ACTING ON BEAMS.

FIG. 5.4 HORIZONTAL FORCES ACTING ON EDGE PANELS.
(a) **Restriction of Lateral Bowing of Beams**

Lateral bowing of the supporting beams due to the assumed membrane action will occur only in the beams marked AB, CD, EF and GH in Fig.5.3. The remaining interior beams of the floor are loaded by equal but opposite horizontal forces (assuming that the panels are of the same thickness) and hence will not bow. The edge beams of the floor will also bow outwards, but since membrane action in the exterior panels in the direction normal to the outside edge has been neglected this bowing is of no concern. The tendency for the beams AB, CD, EF and GH to bow will be restricted by the deep beam action of the adjoining exterior panels. The interior panels which apply the membrane force will also act as deep beams and will thus help to restrict the bowing. Most slab and beam floors are proportioned with exterior and interior panels of approximately the same dimensions. If all the panels are square there is no doubt that the exterior panels will have sufficient lateral stiffness to reduce the bowing of the supporting beams on their interior edges to negligible quantities, since they will act as deep beams with a span/depth ratio of unity. If, however, the panels are rectangular \(L_y > L_x\) the very large horizontal force exerted on the long side \(L_y\) by an interior panel will have to be
restrained by the exterior panel acting as a deep beam with a span/depth ratio of $\frac{L}{L_x}$. As the $\frac{L}{L_x}$ ratio of the panels increases the lateral stiffness of the exterior panel decreases and a stage will be reached where the exterior panels are too narrow to restrict the bowing. The maximum $\frac{L}{L_x}$ ratio of an exterior panel which could be supposed to adequately restrain lateral bowing could be found by determining the displacements by deep beam theory. However, most panels of slab and beam floors are not too rectangular and it is suggested that the resistance to lateral bowing will always be sufficient providing that the $\frac{L}{L_x}$ ratio of the exterior panel is unity, and that $\frac{L}{L_x}$ ratios of as great as 1.5 should not result in too large a reduction in membrane action.

(b) Restriction of the Extension of Beams

Figure 5.4 shows the horizontal forces acting on the interior beam of the exterior panels of one side of the floor. The beam is that marked AB in Fig.5.3 and is one of the beams which undergoes lateral bowing due to membrane forces acting on one side only. The membrane force acting laterally on the beam is $N_x$ (per unit length) and this is to be resisted by tensile forces $P_x$ provided by the intersecting beams acting as ties. Each tie force $P_x$ is
approximately equal to the sum of the membrane forces, $N_x$, acting on the adjacent halves of the panels. In the symmetrical case considered therefore:

$$P_x = L_y \times \text{Mean } N_x.$$  \hspace{1cm} (5.1)

Similarly for the beams running in the $y$ direction the tie forces are:

$$P_y = L_x \times \text{Mean } N_y.$$ \hspace{1cm} (5.2)

Hence the beams will require reinforcement in addition to that placed for ordinary bending in order to provide for the tie forces. It is evident that the stretch of the tie reinforcement due to the forces $P$ will allow the edges of the panels to move outwards slightly. The reduction in ultimate strength due to small outward movements of the edges can be large for thin panels but is negligible for thick panels, as is shown by Figs. 3.4 and 3.5. It is evident, however, that for all thicknesses, the stress in the tie steel should be less than the yield stress when the ultimate load of the panel is reached, otherwise unlimited extension of the beams could occur. To balance the compressive membrane
forces of the slab the tie steel should be placed along the top of each beam. The change in length of the tie steel is difficult to estimate since it is also subjected to strains due to the beam bending in the vertical plane. Thus the strain distribution along the beam due to the tie force and bending has to be established before the extension of the tie steel can be estimated.

(c) **Strength for Lateral Forces.**

The strength of beams acting as ties will be adequate since the requirements of limited extension will ensure stresses of less than the yield stress in the tie steel. The strength required of exterior panels to act as deep beams has also to be considered. It is evident that deep beam action will induce tensile stress in each exterior panel at the middle of the outside edge and at the ends of the interior edge. The tie steel for membrane forces in the other direction, however, should be adequate to cope with this tension.

5.3 **Prerequisites for Tensile Membrane Action in Slab and Beam Floors.**

When the floor is loaded beyond the ultimate flexural load to the stage of tensile membrane action at high deflections of the panels, the directions of the membrane
forces acting on the beams in Figs. 5.3 and 5.4 will be reversed and the supporting beams will act as struts rather than ties. Since the tensile membrane forces will usually be smaller than the compressive membrane forces developed earlier, the requirements of the compressive membrane stage should satisfy the tensile membrane stage. The strut action of the beams will be developed by some of the concrete in the beams going into compression. Hence no additional reinforcement need be placed in the beams for this stage of loading.

5.4 A Study of Economy: Reinforcement Contents of Slab and Beam Floors Resulting From Design With and Without Compressive Membrane Action.

Section 5.2 indicated that although sufficient lateral stiffness from surrounding panels is available in most slab and beam floors to enforce compressive membrane action, extra reinforcement must be placed in the beams to enable them to act as ties resisting the membrane forces. Hence, while use of compressive membrane action allows the designer to reduce the steel contents of panels to less than that required by Johansen's yield-line theory (Johansen neglects membrane action), the steel contents of the supporting beams will be greater than that required for beams supporting
panels designed by Johansen's theory. For economical use of compressive membrane action, therefore, the resulting reduction in steel content of the panels must be greater than the extra reinforcement placed in the beams. The difference in steel contents of panels and beams designed with and without compressive membrane action will now be examined.

Consider a square interior panel of side L of a slab and beam floor. Let the top and bottom steel be placed at 0.15d from the top and bottom faces of the panel, respectively. For simplicity, the steel in compression will be assumed to make no contribution to the flexural strength of the panel. This will be almost exactly true for the case without compressive membrane action, but for the case with compressive membrane action it will mean that the flexural strength is slightly under-estimated. Also, the top and bottom tension steel will be assumed to have equal yield forces, per unit width, i.e., \( T' = T \).

(a) Panel Designed As If Laterally Unrestrained.

Let \( T_u \) be the yield force, per unit width, of the hogging and sagging moment tension steel in both directions. Since the depth of compressed concrete is small in this case the lever arm of the internal couple may be shown to be
approximately equal to 0.95 of the effective depth, and the simple yield moments are:

\[ m = m' = T_u \times 0.95 \times 0.85d = 0.81T_u d. \]

The ultimate flexural load (uniformly distributed) given by Johansen's yield-line theory for this case is:

\[ w_J = \frac{24}{L^2} (m' + m) \]

\[ \therefore \quad \frac{w_J}{24} \left( \frac{L}{d} \right)^2 = 1.62 \frac{T_u}{d} \quad (5.3) \]

The top steel of such a panel may be curtailed at 0.15L from each support (this is sufficient distance to ensure that an alternative collapse mechanism will not form, as is shown in Appendix A), and the cross-sectional area of top and bottom steel, per unit width, is \( \frac{T_u}{f'_{Y}^{s}} \), where \( f'_{Y} \) is the yield stress. Hence the volume of steel in the panel is:

\[ = 2 \frac{T_u}{f'_{Y}^{s}} L \times 1.3L = 2.6 \frac{T_u}{f'_{Y}^{s}} L^2 \quad (5.4) \]

(b) **Panel Designed As If Fully Restrained Laterally**

Let \( T_r \) be the yield force, per unit width, of hogging and sagging moment tension steel in both directions. The ultimate
flexural load given by rigid-plastic theory for this case is, from (2.77), neglecting axial strains and lateral displacements:

\[
\frac{w_u}{24} \left( \frac{a}{d} \right)^2 = k_1 k_3 u (0.333 - 0.386 k_2) + 0.95 \frac{T_r}{d} \quad (5.5)
\]

The strength of the panel is more dependent upon the strength of the concrete than in the last case, and it is estimated that the top steel may be curtailed at 0.1L from each support. The volume of steel in the panel is:

\[
= 2 \frac{T_r}{f_s} L \times 1.2L = 2.4 \frac{T_r}{f_s} L^2 \quad (5.6)
\]

To compare the steel contents of the two designs, consider the panels to have the same ultimate flexure loads and L/d ratios. The volume of concrete used in each case will be the same. Then (5.3) = (5.5).

\[
\therefore 1.62 \frac{T_u}{d} = k_1 k_3 u (0.333 - 0.386 k_2) + 0.95 \frac{T_r}{d}
\]

\[
\therefore T_u = k_1 k_3 u d (0.206 - 0.238 k_2) + 0.586 T_r \quad (5.7)
\]
Saving in volume of steel in the panel allowed by compressive membrane action

\[\text{\( = (5.4) - (5.6)\)}\]

\[\text{\( = \frac{L^2}{f_s} (2.6T_u - 2.4T_r)\)}\]

\[\text{\( = \frac{L^2}{f_s} \left[ k_1 k_3 u d (0.536 - 0.619 k_3) - 0.88 T_r \right]\)}\]  \hspace{1cm} (5.8)

where \(T_u\) was substituted from (5.7).

Now the mean membrane force acting on a supporting beam as a result of compressive membrane action is, from (3.45) or (3.46), neglecting axial strains and lateral displacements,

\[\text{Mean } N = 0.437 k_1 k_3 u d - T_r\]

Hence the tensile force to be provided for the panel by the ties in each of the four surrounding supporting beams is

\[\text{\( = 0.5L \times \text{Mean } N\)}\]

\[\text{\( = 0.219 k_1 k_3 u L d - 0.5T_r L\)}\]
Volume of steel required for ties in the supporting beams for compressive membrane action (the ties are assumed to run along the whole length of each beam), if the steel stress in the ties is \( f_s' \), is

\[
= \frac{L}{f_s'} (0.219 k_1 k_3 u L_d - 0.5 T_r L)
\]

\[
= \frac{L^2}{f_s} (0.876 k_1 k_3 u d - 2 T_r)
\]

(5.9)

The design with membrane action will be more economical than the design without if (5.8)>(5.9). To illustrate the relative magnitudes of the steel contents involved let the concrete have a cube strength of 3,000 lb./sq.in. (hence \( k_1, k_3 = 0.67 \) and \( k_2 = 0.47 \)), and the reinforcement have a yield stress of \( f_y' = 40,000 \text{ lb./sq.in.} \) in the panel and a stress of \( f_y = 40,000 \text{ lb./sq.in.} \) in the beams. Then the volumes of steel for two reinforcement contents are compared below:

<table>
<thead>
<tr>
<th>% of Reinforcement in the Laterally Restrained Panel</th>
<th>Volume of Steel in the Laterally Restrained Panel from (5.6) ( x L^2 d )</th>
<th>Saving in Volume of Steel in Panel, from (5.8) ( x L^2 d )</th>
<th>Volume of Steel Required for Ties, from (5.9) ( x L^2 d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0.0036</td>
<td>0.0110</td>
<td>0.0410</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0240</td>
<td>0.0035</td>
<td>0.0240</td>
</tr>
</tbody>
</table>
It can be seen that if a panel is designed as if fully restrained laterally the volume of steel required for ties in the supporting beams as a result of compressive membrane action will always exceed the volume of steel saved in the panel. The difference in the volumes of steel is surprisingly large but it should be noted that in the comparison full membrane action was assumed for the laterally restrained panel. This is reasonable for thick slabs, but for thinner slabs the difference in the volumes of steel will not be so great due to the reduced membrane action caused by axial strains in the panel and extension of the tie steel. The difference in the volume of steel resulting from the two designs can be explained as follows. Figure 5.5 shows a section through a part of a panel at ultimate load. For simplicity the forces in the top and bottom tension steel are assumed to be equal and the forces in the compression steel are assumed to be zero. It can be seen that the sum of the moments of the internal actions of the part of the panel may be written as

\[ T_g + C_c h \quad (5.10) \]

Now (5.10) is directly proportional to the ultimate strength
Collapse mechanism

**FIG. 5.5** INTERNAL FORCES AT YIELD LINES OF PANEL.

**FIG. 5.6** OCKLESTON'S SLAB AND BEAM FLOOR.

**FIG. 5.7** UNIVERSITY OF ILLINOIS' SLAB AND BEAM FLOOR.
of the panel. In the laterally unrestrained panel, $C_c = T$ at each section since there is no membrane force. In the laterally restrained panel the reinforcement content may be reduced since the concrete force $C_c$ is much higher than in the unrestrained case. However, since $h$ is much smaller than $g$, (5.10) shows that the increase in $C_c$ has to be much greater than the decrease allowed in $T$, if the ultimate loads of the laterally restrained and unrestrained panels are to be the same. The resulting difference between $C_c$ and $T$ is the membrane force $N$, which has to be carried by tie steel in the beams. Since the membrane force $N$ is greater than the allowable decrease in the steel force in the panel, the tie steel required in the supporting beams to resist the membrane forces will be of greater volume than the steel saved in the panel as a result of compressive membrane action.

Thus it is evident that a slab and beam floor cannot be designed economically for full membrane action in all panels when the whole floor is loaded so that all panels collapse at the same load, since more steel is required in the beams to act as ties than can be saved in the panels. This would appear to make compressive membrane action of academic interest only. Where compressive membrane action theory is of use, however, is in the calculation of the ultimate strength
of a panel which is loaded to failure with the adjacent panels only lightly loaded. The strength of the steel in the adjacent panels and beams which is not being used to carry the light loading on those panels is then available to develop compressive membrane action in the heavily loaded panel. By this method a floor which is designed by conventional Johansen's yield-line theory to carry a particular working load on all the panels may be allowed to carry on some panels a working load of many times the design working load, provided that the surrounding panels are lightly loaded.

5.5 Analysis of the Experimental Results of Ockleston and the University of Illinois.

5.5.1 Ockleston's Tests

The tests conducted by Ockleston\(^3,4\) were on three interior panels of a full scale reinforced concrete slab and beam floor. The structural layout of the floor is shown in Fig.5.6. Two test runs were made. The panel marked A was loaded uniformly to failure alone. In the other test run the two adjacent panels marked B1 and B2 were loaded simultaneously to failure. Each test run took two or three days to complete. The panels were very lightly reinforced with steel contents varying between 0.09% and 0.20% of the gross cross-sectional
area of the concrete. Mild steel reinforcement was used. The panels were designed for a working load of 50 lb./sq.ft. live load and 55 lb./sq.ft. dead load. The supporting beams were heavily reinforced, the area of steel exceeding that required for the design floor loading by approximately 50%. The upper surface of the structural concrete of each panel was finished with a layer of mortar screed.

Two features of the panels make the calculation of the theoretical ultimate strengths difficult. The first concerns the mortar screed. It is uncertain whether the bond between the screed and the structural concrete was sufficient for the total thickness of the panels to be considered as structurally effective. Also the mean compressive strength of the screed was 85% of that for the structural concrete. In the calculations to follow the panels will be analysed firstly as if the screed was entirely ineffective structurally, and secondly as if it was fully effective and of the same strength as the structural concrete. The second difficulty concerns the reinforcement. The reinforcement was placed only in the middle strips of the panels, and hence was not placed uniformly across the panels as is assumed in the theory. The panels will be analysed as if the bars that were in the middle strips had been spread uniformly across the panels. The compression steel
will not be considered to contribute to the strength of the panels, since the top steel did not extend over the whole area of the panels and the bottom steel did not extend far enough into the supporting beams for it to be considered with the hogging bending moments. Also for panels A and B1 the amounts of tension steel at opposite short supports was different, but the mean value will be used.

On the basis of the above assumptions the properties of the panels were:

Spans:  
A, B1: \( L_x = 162 \text{in.} \), \( L_y = 190.5 \text{in.} \).
B2 : \( L_x = 162 \text{in.} \), \( L_y = 191 \text{in.} \).

Thickness: \( d = 4.55 \text{in.} \) without screed, or \( d = 5.30 \text{in.} \) with screed.

Concrete: \( u = 4200 \text{ lb./sq.in.} \) (Hence \( k_1, k_3 = 0.61, k_2 = 0.46 \))

Reinforcement:

Yield forces: \( T_x = 450 \text{ lb./in.} , T'_x = 486 \text{ lb./in.} \),
\( T_y = 245 \text{ lb./in.} , T'_y = 218 \text{ lb./in.} \).
* For panel B2, \( T'_y = 291 \text{ lb./in.} \).

<table>
<thead>
<tr>
<th>Steel Depths</th>
<th>( d_x \text{ in.} )</th>
<th>( d'_x \text{ in.} )</th>
<th>( d_y \text{ in.} )</th>
<th>( d'_y \text{ in.} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>With Screed</td>
<td>4.56</td>
<td>2.71</td>
<td>4.29</td>
<td>2.99</td>
</tr>
<tr>
<td>Without Screed</td>
<td>3.81</td>
<td>2.71</td>
<td>3.54</td>
<td>2.99</td>
</tr>
</tbody>
</table>
Table 5.1: Ultimate Flexural Loads of Ockleston's Panels

<table>
<thead>
<tr>
<th>Panel Mark</th>
<th>Actual Load at Failure ( w_{test} ) lb/sq.ft</th>
<th>Johansen. ( \frac{w_{test}}{w_J} )</th>
<th>A Long Edge Laterally Unrestrained. ( \frac{w_{test}}{w_{theory}} )</th>
<th>All Edges ** Laterally Restrained. ( \frac{w_{test}}{w_{theory}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Without Screed</td>
<td>With Screed</td>
<td>Without Screed</td>
<td>With Screed</td>
</tr>
<tr>
<td>A</td>
<td>753</td>
<td>2.90</td>
<td>2.60</td>
<td>1.26</td>
</tr>
<tr>
<td>B1</td>
<td>843</td>
<td>3.26</td>
<td>2.90</td>
<td>1.41</td>
</tr>
<tr>
<td>B2</td>
<td>843</td>
<td>3.14</td>
<td>2.81</td>
<td>1.40</td>
</tr>
</tbody>
</table>

* \( w_{theory} \) from (2.51), (2.52) and (2.53), with \( \Delta u = 0.4d \).

** \( w_{theory} \) from (2.77).

Table 5.1 shows that at failure the theoretical Johansen ultimate loads were well exceeded, whether the screed is taken into account or not. The rigid-plastic theory ultimate loads assuming all edges to be rigidly restrained against lateral displacement (given by 2.77), however, were not reached, even if the screed is considered to be entirely ineffective. One explanation of this is the lack of adequate stiffness of the surrounding panels. As Fig. 5.6 shows, the edge panels of the floor were narrow and the width of an edge panel (6ft.) was only 38% of the length of the adjacent side.
of the interior panel (15ft. 11in.). Thus lateral bowing of these narrow panels under the horizontal membrane forces was possible. If it is assumed that the compressive membrane forces were unable to develop in the direction at right angles to the narrow edge panels then only "one-way" membrane action will be present and (2.51), (2.52) and (2.53) with \( \Delta u = 0.4d \) give the minimum theoretical ultimate loads. It can be seen from Table 5.1 that the experimental ultimate loads fall between the theoretical values for "one-way" membrane action and for full membrane action. This kind of result is to be expected for interior panels with narrow edge panels in one direction. A further explanation for the lack of full membrane action could be inadequate reinforcement in the surrounds to resist the large horizontal forces exerted. This could have resulted in large extensions of the beams which allowed outward movement of the edges of the panels. The crack patterns shown in Ockleston's report indicate that for the two-panel test (panels B1 and B2) there was extensive cracking in the beams and surrounding panels, but for the single panel test (panel A) the cracking in the beams was not extensive. Figure 3.4 illustrates, however, that quite small extensions of the supporting beams will reduce considerably the ultimate load of panels with the
span/depth ratio considered. Thus lack of lateral stiffness of the surround in one direction and extension of the beams caused the panels not to reach the theoretical "fully restrained" ultimate load.

The deflections of the panels at ultimate load are of interest since in the theory an empirical value for the central deflection is used. In the single panel test (panel A) at ultimate load, relative to the corners of the panel, the central deflection of the panel was $2\tfrac{1}{2}$ in. and the central deflections of the supporting beams were $\tfrac{1}{4}$ in. to $\tfrac{1}{2}$ in. In the two panel test (panels B1 and B2) at ultimate load, relative to the corners of the panel, the central deflections of the panels were 2 in. and the central deflections of the supporting beams were $\tfrac{1}{4}$ in., except for the beam between the two panels which was $\tfrac{3}{4}$ in. Thus at ultimate load the central deflections of the panels relative to the centres of the supporting beams were approximately 0.3 to 0.4 of the total thickness of the panels, which is in the range found in the single panel tests reported in Chapter 2.

Since the tests were stopped once the ultimate flexural strengths of the panels were reached the behaviour in the tensile membrane stage was not investigated.

5.5.2 The University of Illinois' Interior Panel Tests.

The tests conducted by the University of Illinois were on a $\tfrac{1}{4}$ scale model of a nine panel reinforced concrete
slab and beam floor. The floor had been designed according to the American Concrete Institute Building Code requirements for a working load of 70 lb./sq.ft. live load and 75 lb./sq.ft. dead load. The structural layout of the floor is shown in Fig. 5.7. The reinforcement of the panels was of annealed mild steel and was uniformly spread over the whole width. In the calculations the compression steel will not be considered to contribute to the strength of the panels, since the top steel did not extend over the whole area of the slabs and the bottom steel did not extend far enough into the supporting beams for it to be considered with the hogging bending moments. The strength of the concrete was measured by 2in. diameter x 4in. cylinders and for the purpose of analysis the cylinder strength will be taken to be 0.8 of the cube strength. The properties of the interior panel were:

Spans: \( L_x = L_y = 57 \text{in.} \)

Thickness: \( d = 1.5\text{in.} \)

Concrete: \( u = 2610/0.8 = 3260 \text{ lb./sq.in.} \) (Hence \( k_1 k_3 = 0.66, \ k_2 = 0.47 \)).

Reinforcement:

Yield Forces: \( T_x' = T_y' = 184 \text{ lb./in.} \)
\( T_x = T_y = 161 \text{ lb./in.} \)

Mean steel depths: \( d_x' = d_y' = 1.22\text{in.} \)
\( d_x = d_y = 1.19\text{in.} \)
A total of 37 loading tests were conducted on the floor before two tests to failure were conducted. In the first test to failure the whole floor was uniformly loaded. The ultimate load was reached at 537 lb./sq.ft. when the beam-column joints along one side of the floor failed. At this load, however, yielding was general and was observed at every critical section of the structure. The collapse mechanism was of the type shown in Fig.5.2a. The interior panel had undergone the least damage and a second load test to failure was conducted. In this test 250 lb./sq.ft. was applied to each of the corner panels, to hold the edges of the floor down, and the interior panel alone was loaded uniformly to collapse. Failure occurred at 829 lb./sq.ft. when two of the supporting beams of the interior panel failed in combined bending, torsion and shear.

The load carried by the interior panel at the stage of collapse of the supporting beams (829 lb./sq.ft.) was 1.94 times the theoretical Johansen ultimate load and 0.81 of the rigid-plastic theory ultimate load assuming all edges to be fully restrained against displacement given by (2.77). When the supporting beams failed, however, the load-central deflection curve for the interior panel was still rising, and it is evident that had the supporting beams not failed
the load carried by the panel may have reached the theoretical load given by (2.77). Relative to the corners, at maximum load, the central deflection of the interior panel was 2\(\frac{1}{2}\)in., and the central deflections of the supporting beams were 0.2in. Hence relative to the centre of the beams the central deflection of the panel was 1.3 times the slab thickness (1.3d). This is much greater than the maximum value of 0.5d found in the single panel tests of Chapter 2. It is apparent that the large deflection was due to the residual deflections of the 38 loading tests which preceded the test to failure of the interior panel. The central deflection at failure of 1.3d included a residual central deflection of 0.5d which was present before the panel was loaded to failure. The central deflection was large enough to cause tensile membrane action (as indicated by cracks through the full depth of the concrete) in the central region of the panel at ultimate load, but it is apparent that the edge regions of the panel were carrying load by compressive membrane action.

It is also of interest to note that in the first test run to failure (in which all panels were loaded uniformly) the ultimate load was approximately 1\(\frac{1}{2}\) times the Johansen load for collapse confined to the panels, and the
mechanism of failure which developed was the folding type of Fig. 5.2a. Thus sufficient membrane action was available to enhance the strength of even the corner panels to the extent of causing the simple folding collapse mechanism (which is not helped by membrane action) to occur.

5.6 Small-Scale Tests on the Lateral Stiffness and Strength of Exterior Panels

5.6.1 The Slabs

Slab series F and G were made to investigate the lateral stiffness and strength of exterior panels of slab and beam floors. Each slab was an idealized nine panel continuous slab and beam floor with panels arranged three by three. The interior panel of each floor was 12in. square and the surrounding eight exterior panels were either 12in. square or smaller. The interior panels of slab series F and G had span/depth ratios of approximately 30 and 18, respectively. In all cases the interior panel of the floor was loaded uniformly to failure by external loading acting normal to the plane of the slab. The exterior panels were not loaded externally but were subjected to the loading caused by the compressive membrane action induced in the interior panel. Supporting beams were not cast with the floor since the panels were
supported by rollers bearing against the test frame along the lines of the edges of the interior panel and along the outside edges of the floor. Figure 5.8 shows the area of loading and the positions of the supporting rollers of the slabs. The rollers were placed above or below the slabs depending upon the direction of the reaction. The interior panels of all slabs were unreinforced in order to make the compressive membrane stresses within the panel a maximum. In most cases, however, reinforcement was placed around the edges of the panels to simulate tie reinforcement in the supporting beams. Each type of slab was cast in duplicate.

(a) Slab Series F

The 14 slabs of this series were of overall dimensions 36in. x 36in. and were models of floors consisting of nine 12in. x 12in. panels. Since each exterior panel of the idealized floor was square and of the same dimensions as the interior panel, it was anticipated that the exterior panels would have sufficient lateral stiffness to prevent lateral bowing around the edges of the interior panel due to membrane forces acting outwards. The slabs were tested mainly to investigate the requirements for strength of the exterior panels when resisting horizontal forces, and
Loaded area (interior panel).

0.5" dia. roller supports.

Plan.

Section AA

FIG. 5.8 LOADING AND SUPPORTING OF SLABS OF SERIES F & G.
also to investigate the effect of extension of the tie reinforcement around the edges of the interior panel. The positions of the supporting rollers and of the reinforcement bars are shown in Fig. 5.9. The reinforcement was of 1/8in. square bright steel bar placed at the mid-depth of the slab. The "square ring" reinforcement was joined at the laps by welding or some other fusion process. Details of the properties of the steel and the mortar used for the slabs are given in Appendix C. Details of the reinforcement positions in the exterior panels, as shown in Fig. 5.9, were:

Slabs F1, F2 : Unreinforced.
Slabs F3, F4 : One 8in. long bar at 45° to the spans placed 1lin. from each corner of the interior panel.
Slabs F5, F6 : As for F3 and F4, with the addition of one 12in. long bar at the middle of, and 1lin. from, each outside edge.
Slabs F7, F8 : One square ring bar with 14in. side around the interior panel, and one 12in. long bar at the middle of, and 1lin. from, each outside edge.
Slabs F9, F10 : One square ring bar with 14in. side around the interior panel, and one square ring bar with 34in. side around the outside edge.
All slabs: Interior panel 12"x12"
Overall size 36"x36"

--- Roller supports.
--- 0.125" square bar reinforcement at mid-depth.
--- Type of crack in exterior panel at ultimate load.

FIG. 5.9 SLAB SERIES F : POSITION OF SUPPORT ROLLERS, REINFORCEMENT AND CRACKS IN EXTERIOR PANELS.
Slabs F11, F12: Two square ring bars, with 14in. and 15in. sides, respectively, around the interior panel, and one square ring bar with 34in. side around the outside edges.

Slabs F13, F14: Three square ring bars, with 14in., 15in. and 16in. sides, respectively, around the interior panel, and one square ring bar with 34in. side around the outside edge.

(b) Slab Series G

The six slabs of this series were of overall dimensions of either 36in. x 36in., 30in. x 30in. or 24in. x 24in., and were models of floors with a 12in. x 12in. interior panel and the same size, or smaller, exterior panels. The slabs were tested mainly to investigate the effect of lateral bowing at the edges of the interior panel due to membrane forces acting outwards on to exterior panels of smaller width, and also to investigate the effect of extension of the tie reinforcement around the edges of the interior panel for slabs with a smaller span/depth ratio than for series F. The positions of the supporting rollers and of the reinforcement bars are shown in Fig:5.10. The reinforcement was of ¼in. square bright steel bar placed at the
All slabs: Interior panel 12"x12"

- Roller supports.

0.25" square bar reinforcement at mid-depth.

Type of crack in exterior panel at ultimate load.

FIG. 5.10 SLAB SERIES G: POSITION OF SUPPORT ROLLERS, REINFORCEMENT AND CRACKS IN EXTERIOR PANELS.
mid-depth of the slab. The "square-ring" reinforcement was joined at the laps by welding or by some other fusion process. Details of the properties of the mortar used for the slabs are given in Appendix C. Details of the reinforcement positions in the exterior panels, as shown in Fig. 5.10, were:

Slabs G1, G2: Two square ring bars, with 14in. and 16in. sides, respectively, around the interior panel, and one square ring bar with 34in. side around the outside edges.

Slabs G3, G4: As for G1 and G2, except that the outside ring bar had a 28in. side.

Slabs G5, G6: As for G1 and G2, except that the outside ring bar had a 22in. side.

5.6.2. The Loading Arrangements

Full details of the loading frame are given in Appendix C. Figure 5.8 shows diagramatically the method of loading and supporting. The rollers allowed horizontal movement to occur, but prevented vertical movement along the line of support.

The load was applied to the slabs in increments in order to measure deflections and observe the development of
cracking. In most cases the test runs from zero to ultimate load took approximately one half of an hour.

5.6.3 The Axial Strains and Lateral Edge Movements of Square Interior Panels Under Short-Term Loading With Continuous Tie Steel Around the Edges.

To determine the theoretical axial strains and lateral edge displacements for the slabs of series F and G which had square ring steel around the interior panel it will be assumed that all the "ring tension" in the surrounding panels was taken by the ring steel, and that the ring steel was stressed uniformly along its length. The lateral movement of the edges of the interior panel will be due to the stretch of the ring steel. Also, the loading is short-term and hence only elastic axial strains will occur in the concrete of the interior panel. The theory required has been developed in Chapter 3 and will now be rewritten in the form required in the calculations for the test slabs.

The interior panels were square and the mean membrane force, per unit width, acting in the direction of each span of the interior panels is, from (3.45):

\[
\text{Mean } N = k_1 k_3 u d \left\{ \frac{7}{16} - \frac{1}{4} (\varepsilon_e + \frac{2t}{L}) (\frac{K}{d})^2 \right\}
\]

(5.11)
where $L = \text{Span of interior panel}$

$t = \text{Outward movement of each edge.}$

$\varepsilon_e = \text{Elastic axial strain in panel.}$

:. The elastic axial strain in the panel is:

$$\varepsilon_e = \frac{\text{Mean N}}{E_c \, d} = \frac{k_1 k_3 u}{E_c} \left\{ \frac{7}{16} - \frac{1}{4} \left( \varepsilon_e + \frac{2t}{L} \left( \frac{L}{d} \right)^2 \right) \right\}$$

$$= k_1 k_3 u \left( \frac{7d^2 - 8Lt}{16E_c d^2 + 4k_1 k_3 uL^2} \right) \quad (5.12)$$

If $A_t$ is the cross-sectional area of the tie steel along one side of the panel, the stress in the tie steel due to membrane force is:

$$= L \times \frac{\text{Mean N}}{A_t}$$

$$= \frac{k_1 k_3 u d L}{2 A_t} \left\{ \frac{7}{16} - \frac{1}{4} \left( \varepsilon_e + \frac{2t}{L} \left( \frac{L}{d} \right)^2 \right) \right\} \quad (5.13)$$

:. Outward movement of each edge of the panel due to stretch of the tie steel is

$$t = \frac{k_1 k_3 u d L L}{4 A_t E_s} \left\{ \frac{7}{16} - \frac{1}{4} \left( \varepsilon_e + \frac{2t}{L} \left( \frac{L}{d} \right)^2 \right) \right\} \quad (5.14)$$
where $L_t =$ Length of tie steel along one side of the interior panel (= length of one side of the ring steel).

On substituting $\varepsilon_e$ from (5.12) into (5.14) and rearranging, the following expression for $t$ results:

$$
t = \frac{1}{16A_t E_s (4E_c d^2 + k_1 k_3 u L^2)} \left( \frac{8L}{7E_c L L_t d^3 k_1 k_3 u} + \frac{8L}{7d^2} \right)
$$

(5.15)

The values of $\varepsilon_e$ and $t$ may be found from (5.12) and (5.15). The values of $\varepsilon_e + \frac{2t}{L}$, which are required for the calculation of theoretical ultimate loads, may then be calculated.

5.6.4 Discussion of Test Results and Theory

The quantity of tie steel placed around the edges of the interior panels of the first few slabs of series F was deliberately made small in order to investigate the modes of failure of exterior panels when under lateral loading from the membrane forces of the interior panels. In the other slabs of series F and in all the slabs of series G sufficient tie steel was placed to provide tensile resistance when the concrete cracked. The modes of failure and types of cracking found in the exterior panels during the tests...
are shown diagrammatically in Figs. 5.9 and 5.10, and in the plates of Figs. 5.11 and 5.12. The yield-line patterns of the interior panels are also shown in the plates. Theoretically the interior yield lines should pass along the diagonals. In most cases, however, the rectangular panel type of pattern developed with a small length of yield line at the centre running parallel to one pair of the edges. This small length of yield line always took the direction of the first crack to form at the centre. The difference in the theoretical ultimate loads given by this unsymmetrical pattern and the assumed diagonal pattern, however, is negligible.

Slabs F1 and F2 were completely unreinforced and reached maximum load when cracking from the diagonal yield lines of the interior panel extended through the exterior panels and caused complete separation of the segments of the yield-line pattern. In slabs F3 and F4 reinforcement bars were placed across the corners to provide tensile resistance across such extended diagonal cracking. The maximum load on the interior panel of these two slabs was reached when a crack opened at the middle of the outside edge of the exterior panels due to tension caused by deep beam action. As a result of this type of failure slabs
Note: White square indicates edges of interior panel.

FIG. 5.11 UNLOADED FACES OF SLAB SERIES F AT END OF TEST.
Slab G1
(similar to G2)

Slab G3
(similar to G4)

Slab G6
(similar to G5)

Note: White square indicates edges of interior panel.

FIG. 5.12 UNLOADED FACES OF SLAB SERIES G AT END OF TEST.
F5 and F6 were reinforced by bars along the middle of the outside edges of the exterior panels as well as across the corners. Failure of these slabs occurred, however, when cracks from the diagonal yield lines of the interior panels extended around the ends of the corner bars and out to the corners of the floor. To control this type of cracking in F7 and F8 a continuous ring bar was placed around the interior panel and bars along the middle of the outside edges of the exterior panels. These two slabs reached maximum load when cracking from the diagonal yield lines of the interior panel extended through the ring bar at the corners and ran out to the unreinforced regions of the outside corners of the floor, where they widened considerably. It was then made clear that cracking of the exterior panels could only be controlled by placing ring bars around the edges of the interior panel and around the outside edges of the floor. Hence ring bars were placed in the remaining slabs of series F and in the slabs of series G. In these slabs at failure cracking extended from the diagonal yield lines of the interior panels through the ring bars but the tensile resistance provided across the cracks restricted the separation of the exterior panels.
Figures 5.13 and 5.14 show the load versus central deflection curves for the interior panels. The slabs of series F reached maximum load when cracking extended into the exterior panels and allowed outward movement of the edges. The load deflection curves of slabs F1 to F6 were not traced beyond maximum load since the load fell off suddenly due to the unrestricted cracking. In the case of slabs F7 to F14 and the slabs of series G, however, the fall off in load when the exterior panels cracked was arrested as the reinforcement across the cracks stretched and took up the tensile force that had been carried by the concrete. The slabs of series G had smaller span/depth ratios and hence were not so much affected by lateral displacements of the edges at cracking. In all slabs the initial deflection of the interior panels which occurred with only a small increase in load at the start of each load-deflection curve was due to the slabs having to deflect slightly when the load was first applied before coming into contact with the rollers around the edges of the interior panels. The load-deflection curves show that the interior panels of the slabs with adequately reinforced exterior panels reached ultimate load at a central deflection varying between 0.25 and 0.5 of the slab thickness. The small...
FIG. 5.13 LOAD-DEFLECTION CURVES: SLAB SERIES F.
Vertical uniform loading on interior panel.

lb./sq.in.

Scale: 0 0.1 in.

Vertical deflection at centre of interior panel.

FIG. 5.14 LOAD-DEFLECTION CURVES: SLAB SERIES G.
deflections of some interior panels at failure was due to the premature failure caused by the cracking of exterior panels.

Table 5.2 sets out the experimental results and compares the maximum load carried by the interior panels with theory. The values of $k_1$, $k_3$, and $k_a$ required by the theory were obtained from Fig. 2.2. The theoretical rigid-plastic ultimate loads calculated assuming no lateral movement of the edges were never reached by the slabs of series F, but were reached by some of the slabs of series G. This is to be expected since the span/depth ratio of the slabs of series F was approximately 30 and hence the ultimate strength of these slabs could be considerably reduced by lateral edge movements due to cracking and by axial strains. The slabs of series G were considerably thicker, with span/depth ratios of approximately 18, and hence were not appreciably affected by small movements and strains. Table 5.2 shows that the ratio of the maximum load from test to the theoretical rigid-plastic ultimate load was of the same order for the reinforced and unreinforced slabs of series F, illustrating that the reinforcement was only fully utilized after cracking of the exterior
Table 5.2: Uniformly Loaded Interior Panel of a Nine Panel Floor. Slab Series F and G. Comparison of Experiment and Theory.

<table>
<thead>
<tr>
<th>Slab</th>
<th>d (in.)</th>
<th>L (in.)</th>
<th>u (lb/sq. in.)</th>
<th>Max. test</th>
<th>Load.</th>
<th>w_u\text{test} _\text{lb/sq.in.}</th>
<th>Theor.*</th>
<th>Ulit.Load.</th>
<th>fm(2.77)</th>
<th>w_u(2.77)</th>
<th>Theor.**</th>
<th>Ulit.Load.</th>
<th>fm(3.18)</th>
<th>w_u(3.18)</th>
<th>Theor. Stress in Tie Steel</th>
<th>fm(5.13)</th>
<th>w_u(5.13)</th>
<th>w_u\text{test} _\text{lb/sq.in.}</th>
<th>w_u\text{test} _\text{lb/sq.in.}</th>
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<tbody>
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<td>F1</td>
<td>0.354</td>
<td>33.9</td>
<td>4030</td>
<td>6.09</td>
<td>8.15</td>
<td>-</td>
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<tr>
<td>F2</td>
<td>0.357</td>
<td>33.6</td>
<td>3500</td>
<td>5.08</td>
<td>7.33</td>
<td>-</td>
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<tr>
<td>F3</td>
<td>0.369</td>
<td>32.5</td>
<td>3760</td>
<td>4.96</td>
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<tr>
<td>F4</td>
<td>0.379</td>
<td>31.6</td>
<td>4350</td>
<td>7.45</td>
<td>9.95</td>
<td>-</td>
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<tr>
<td>F5</td>
<td>0.388</td>
<td>30.9</td>
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<td>6.45</td>
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<td>F6</td>
<td>0.385</td>
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<td>F7</td>
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<td>F9</td>
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<td>31.7</td>
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<td>5.28</td>
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<td>32,600</td>
<td>0.65</td>
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<tr>
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<td>5.17</td>
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<td>3.90</td>
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<td>F11</td>
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<td>2750</td>
<td>5.19</td>
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<td>17.3</td>
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<td>19.6</td>
<td>19.5</td>
<td>19.5</td>
<td>19,400</td>
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<tr>
<td>G5</td>
<td>0.663</td>
<td>18.1</td>
<td>3250</td>
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<td>23.7</td>
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<tr>
<td>G6</td>
<td>0.640</td>
<td>18.7</td>
<td>2670</td>
<td>16.4</td>
<td>19.2</td>
<td>19.0</td>
<td>19,400</td>
<td>0.85</td>
<td>0.86</td>
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</table>

* Assumes a rigid-plastic material and no lateral movement of the edges.

** Includes axial strains and lateral movement of the edges.
panels had occurred, and that the maximum load carried by the interior panel was dependent upon the tensile strength of the concrete.

Table 5.2 also shows the theoretical ultimate loads calculated from (3.18), with axial strains and lateral edge displacements found from (5.12) and (5.15). In deriving (5.12) and (5.15) it was assumed that the membrane forces of the interior panel were resisted only by the ring bars, and hence the equations apply after the exterior panels have cracked. It was also assumed that the ring bars were uniformly stressed along their whole length. This was reasonable for the test slabs since the ring bars were of bright steel and it is probably that the bond between the steel and concrete was broken down when the concrete cracked. The stresses may have been higher at the bends at each corner of the panels, however. The absence of bond along the length of the ring bars also means that the bars carried only the ring tension and could not carry any of the deep beam stresses. In calculating the axial strains and lateral boundary displacements from (5.12) and (5.15) it was assumed that the membrane forces were resisted only by the ring bar or bars placed at the edges of the interior panel. This neglects the ring tension carried by the ring bar at
the outside edge of the floor which, although necessary to control the cracking there, must be too far away from the interior panel to carry the membrane forces directly. In the calculations of (5.12) and (5.15) it was also assumed that $E_c = 1000 \text{ lb./sq.in.}$ (the value used in the calculations of Chapter 3 for cube strengths of up to 4000 lb./sq.in) and that $E_s = 30 \times 10^6 \text{ lb./sq.in.}$ The theoretical ultimate loads calculated including axial strains and edge displacements should be compared with the loads carried by the interior panels directly after cracking of the exterior panels had occurred, since only then does the tie steel resist the full membrane stress as is assumed in the theory. In Table 5.2 the ratio of the theoretical ultimate load from (3.18) and the maximum load carried by each interior panel is shown. In the case of slabs F11 to F14 and G1 to G6 the agreement between the maximum load carried by the panel and the theoretical ultimate load is reasonable since, as Figs. 5.13 and 5.14 show, the fall off in load carried by the interior panels of those slabs when the exterior panels cracked is small. In the case of slabs F7, F8, F9 and F10 the maximum load carried by the interior panels is much higher than the theoretical ultimate loads, but, as Fig. 5.13 shows, the fall off in load after cracking
of the exterior panels of those slabs is high and if the theoretical loads are compared with the experimental loads carried directly after cracking the agreement is good. Thus the full theory including axial strains and edge displacements gives a good indication of the ultimate loads if the tensile strength of the concrete of the exterior panels is neglected. Table 5.2 also shows the theoretical stress in the tie steel at ultimate load calculated from (5.13). It can be seen that this stress is well below the yield stress in all cases.

Another aspect of interest is the possible reduction in ultimate strength of the test slabs due to lateral bowing of the exterior panels. The theoretical ultimate loads of Table 5.2 were found assuming that the lateral bowing was negligible. This assumption is reasonable for the slabs of series F and slabs G1 and G2 since the exterior panels were of the same dimensions as the interior panels. For slabs G3 to G6, however, the widths of the exterior panels were smaller (see Fig. 5.10), the ratio of the width of the exterior panel to the span of the interior panel being 0.75 for slabs G3 and G4, and 0.50 for slabs G5 and G6. The effect of this is clearly
indicated in Table 5.2 which shows the average ratio of $w_{\text{test}}/w_u(3.18)$ to be 1.11 for slabs G1 and G2, 0.91 for G3 and G4, and 0.80 for G5 and G6. Thus the effect of bowing of narrow exterior panels was considerable, even though the slabs of series G had a low span/depth ratio. On this basis it is recommended that if the effect of bowing is to be excluded the exterior panels should be square.
6.1 General Conclusions

The theoretical and experimental results have already been fully discussed at the end of each chapter. The main conclusions reached are as follows:

(a) The ultimate flexural load of reinforced concrete slabs with boundaries restrained against lateral movement can greatly exceed the ultimate flexural load given by Johansen's yield-line theory. This is due to the increased flexural capacity of the slabs at the yield lines as a result of the compressive membrane forces induced by the boundary restraints. The enhancement in strength is greatest for lightly reinforced slabs and may approach ten times the Johansen loads for slabs with the minimum reinforcement content allowed by the code of practice.

(b) The yield-line theory developed for slabs with rigid boundaries, using a rigid-plastic strip approximation and an empirical value for the central deflection at the ultimate flexural load, gives a good indication of the strength of slabs tested under short-term uniform loading in extremely stiff surrounding frames. The neglect of axial strains is reasonable
for short-term loading since the elastic strains alone are too small to reduce the membrane action significantly. A theory is developed to determine the central deflection at ultimate load which depends on an empirical value for the width of the yield bands, but it is shown that sufficient accuracy results from assuming that the central deflection at ultimate load is either 0.5 of the slab thickness for slabs with all edges fully restrained or 0.4 of the slab thickness for slabs with three edges fully restrained. Approximate yield-line patterns with the corner yield lines at 45° to the slab edges give a good approximation for the ultimate flexural load, even when the boundary restraints produce membrane forces in one direction only, and lead to much simplified expressions. The tests also show that the onset of cracking of slabs is delayed by restraint against lateral displacement at the boundaries.

(c) The generalized yield-line theory developed to include the effects of axial strains in the plane of the slab and small lateral movements at the boundaries shows that these effects can cause a significant reduction in the ultimate flexural strength, especially in the case of thin slabs. The generalized theory leads to lengthy equations and it is better to use the rigid-plastic rigid-boundary theory in conjunction with precalculated reduction coefficients which take into account the reduction in strength due to the axial strains and boundary displacements. The tests show that axial creep strains
which occur during long periods of sustained loading at practical working loads are very small, due to negligible cracking at that load, and that only elastic and shrinkage axial strains need to be considered.

(d) At large deflections after the ultimate flexural load the load carrying capacity of reinforced concrete slabs with laterally restrained boundaries as a tensile membrane is considerable and for high reinforcement contents it can exceed the ultimate flexural strength. The theory developed which assumes that the reinforcement acts as a plastic tensile membrane and neglects the strength of the concrete gives a conservative estimate of the load carried. It is suggested that for mild steel reinforcement a central deflection of 0.1 of the short span should be considered to be the limit of the tensile membrane stage, since beyond that deflection some of the reinforcement bars commence to fracture.

(e) Examination of the stiffness of surrounding panels required to enforce compressive membrane action in loaded interior panels shows that if the effects of lateral edge displacements due to lateral bowing are to be excluded the exterior panels should be square. Also, to enforce membrane action special tie steel should be placed in the supporting beams to resist membrane forces since the tensile strength of
the concrete of the exterior panels is inadequate to carry the large forces involved. This tie steel should be placed continuously around the supporting beams if failure of exterior panels is to be avoided. The stretch of the tie steel will cause outward movements of the slab boundaries which must be included in the calculations for strength of the interior panels. An examination of the reinforcement contents of slab and beam floors shows that more reinforcement is required for floors designed with membrane action than is required for floors designed without membrane action, because more steel is required for ties in the beams than can be saved in the panels. However, the real use of compressive membrane action theory is in showing that floors designed by conventional theory without membrane action can carry working loads on some interior panels which are much higher than the design working loads provided that the surrounding panels are lightly loaded, since the steel which is not being fully utilized in the lightly loaded panels can be used to develop membrane forces in the heavily loaded panels.

6.2 Suggestions for Future Work

There are many aspects of membrane action in slab and beam floors which require investigation. Some of the investigations could be conducted on single panels with
controlled boundary restraints, but most of them involve consideration of the behaviour of the floor as a whole.

(a) **Single Panels**

Some features of the behaviour of laterally restrained single panels which require investigation are:

i) The effect of steel ductility on the stage of fracture of reinforcement bars of uniformly loaded slabs in the tensile membrane stage. The slabs tested by the Author and by Powell\(^5\) were reinforced by mild steel bars. More extensive tests are required, especially for slabs reinforced with cold worked high tensile steel and for slabs with draped reinforcement.

ii) The behaviour of slabs under concentrated loading. The slabs considered in the present programme were uniformly loaded. Some of these slabs showed signs of secondary shear failures after the ultimate flexural load and so it is expected that punching shear failures would intervene before laterally restrained slabs loaded by concentrated loads reached their full enhanced ultimate flexural strength. However, some enhancement of ultimate flexural strength should be possible, (Thomas\(^6\) has reported enhanced loads) and the problem is worth investigating since it applies to wheel loads on interior panels of bridge decks.
(b) Slab and Beam Floors

The investigation into the lateral stiffness and strength of exterior panels and supporting beams reported in Chapter 5 can only be regarded as preliminary. Further work should be conducted on:

i) The extension of tie steel in supporting beams. Knowledge of the magnitude of the extension is necessary for the ultimate flexural strength calculations of interior panels. The extension will be influenced by the bending stresses of the beam, as well as by the membrane forces carried, since the steel will generally be placed in the top of the beam and will be bonded to the concrete.

ii) The effectiveness as tie steel of reinforcement placed in surrounding panels. There is no doubt that tie steel placed in the supporting beams is effective, but the usefulness of steel placed some distance away from the source of membrane force needs to be investigated since such steel will often be available.

iii) The deep beam stresses induced in exterior panels as a result of horizontal loading imposed by the membrane forces of the interior panels. In order to provide reinforcement for the tensile stresses induced by such loading the distribution of stress needs to be determined.
iv) The degree of lateral bowing of exterior panels when loaded by horizontal membrane forces. Deep beam theory is required to establish the amount of lateral deformation which occurs for panels of various span ratios so that the reduction in strength due to outward bowing of various size surrounds can be included in the ultimate strength equations.
Appendix A: Design of Reinforced Concrete Slabs by Yield-Line Theory.

Appendix B: Slab Series A, B, C, D and E: Details of Slabs, Specimens and Testing Arrangements.

B.1 The Slabs

B.1.1 The Concrete Mix

The concrete was made from aggregate composed of \( \frac{3}{8} \) in. graded crushed limestone and Holm's fine sand in the ratio of 2:1, by weight. The grading of the resulting aggregate is shown in Table B.1.

Table B.1. Grading of Aggregate Used for Slabs. Percentage Passing, by Weight.

<table>
<thead>
<tr>
<th>Slab</th>
<th>British Standard Sieve Size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>%in.</td>
</tr>
<tr>
<td>A2, A3</td>
<td>100</td>
</tr>
<tr>
<td>A1, A4, B1, B2, B3, B4</td>
<td>100</td>
</tr>
<tr>
<td>C1, C2, C3, C4, D1, D2, D3</td>
<td>99.2</td>
</tr>
<tr>
<td>D4, D5, E1</td>
<td>98.2</td>
</tr>
<tr>
<td>E2, E3</td>
<td>99.4</td>
</tr>
<tr>
<td>E4, E7, E6</td>
<td>99.5</td>
</tr>
<tr>
<td>E5, E8</td>
<td>98.9</td>
</tr>
</tbody>
</table>

Ordinary Portland cement was used and the aggregate/cement ratio was 6.0, by weight. The water/cement
ratio was in the range 0.6 to 0.65, by weight. The concrete, although short of middle size aggregate particles, was easily compacted.

B.1.2 The Properties of the Steel

The reinforcement was of lead bath annealed mild steel to ensure that strain hardening after yield was kept to a minimum. The reinforcement was delivered in coils and kinked badly when unrolled. The kinking was undesirable since it would have made accurate positioning of the reinforcement in the moulds difficult. In order to straighten the steel it was cut into 240ft. lengths and tensioned to the yield stress. Samples were cut from each length after straightening and tension tests conducted to obtain stress-strain curves. Figure B.1 shows sample stress-strain curves for the four diameters of reinforcement used. It can be seen that in most cases the yield stress was well marked and that the amount of strain hardening which occurred after yielding was small. The yield stresses and the ultimate tensile strengths of the reinforcement used in the slabs are shown in Table B.2. In the few cases where the yield stress was not clearly defined the British Standard (BS.785) definition of the stress at a strain of 0.005 was used.
FIG. B.1 SAMPLE STRESS-STRAIN CURVES FOR MILD STEEL REINFORCEMENT.
### Table B.2: Yield Stresses and Ultimate Tensile Strengths of the Reinforcement.

| Slab | Short Span Steel | | | Long Span Steel | |
|------|------------------|------------------|------------------|
|      | Dia. in. | Yield Stress lb/sq.in. | Ultimate Tensile Strength lb/sq.in. | Dia. in. | Yield Stress lb/sq.in. | Ultimate Tensile Strength lb/sq.in. |
| A1   | 0.128   | 51,000            | 62,000            | 0.128   | 51,000            | 62,000            |
| A2   | 0.188   | 40,000            | 57,000            | 0.128   | 47,000            | 63,000            |
| A3   | 0.245   | 51,000            | 64,000            | 0.128   | 51,000            | 62,000            |
| A4   | 0.313   | 45,000            | 59,000            | 0.128   | 51,000            | 62,000            |
| B1   | 0.128   | 49,000            | 61,000            | 0.128   | 49,000            | 61,000            |
| B2   | 0.188   | 45,000            | 58,000            | 0.128   | 47,000            | 63,000            |
| B3   | 0.245   | 51,000            | -                 | 0.128   | 47,000            | 63,000            |
| B4   | 0.313   | 44,000            | -                 | 0.128   | 44,000            | -                 |
| C1   | 0.128   | 48,000            | 63,000            | 0.128   | 48,000            | 62,000            |
| C2   | 0.188   | 46,000            | 55,000            | 0.128   | 48,000            | 62,000            |
| C3   | 0.245   | 49,000            | 62,000            | 0.128   | 47,000            | 62,000            |
| C4   | 0.313   | 46,000            | 59,000            | 0.128   | 47,000            | 62,000            |

**Note:** The slabs of series D and E were unreinforced.

**B.1.3 The Mould and Casting**

The base of the mould was formed from a sheet of 3/8in. thick mild steel plate stiffened by 4in. x 2in. steel channels on the underside. The sides of the mould were of timber for the reinforced slabs, and of timber or steel angles...
for the unreinforced slabs. All slabs were cast with edge strips for clamping purposes around what was to be the tested area. In cases where the tested region of the slab was to be cast thinner than the edge strips a sheet of plywood with the appropriate thickness and size was fastened to the centre of the base of the mould. All slabs were cast with what was to be the loaded face of the slab to the top of the mould. The holes in the edge strips through which the holding down studs were to pass were formed by attaching bright steel rods covered with P.V.C. sleeving to the base of the mould. The reinforcement in the slabs was continued through the edge strips to the outside edges of the slabs. During casting the reinforcement in the bottom of the slab was held in place and kept straight by mortar blocks and by a slight tension induced by nuts tightened against the sides of the mould. The reinforcement in the top of the slab was propped by mortar blocks, threaded through holes in the sides of the mould and tied to tensioned bars running at right angles. The top bars were tied together at every node point of the reinforcement mesh and the bottom bars were tied together at half the node points. Figures B.2 and B.3 show slabs set up ready for casting.
FIG. B.2 MOULD AND REINFORCEMENT FOR SLAB A3.

FIG. B.3 MOULD AND REINFORCEMENT FOR SLAB C4.
For each slab the concrete was mixed in four batches in a 1½ cubic ft. capacity Cumflow concrete mixer. The concrete was compacted using a vibrator attached to the mould. The mould was mounted on rubber pads. Excess concrete was screeded off using the vibrator attached to a length of channel.

The slabs were cured for 6 to 10 days* after casting by placing damp hessian over the concrete surface.

B.1.4 Slab Dimensions and Reinforcement Contents.

The tested area of each slab was rectangular panel spanning 60in. x 40in. For a fully restrained edge a 12½ in. wide edge strip was provided for clamping purposes. For a simply supported edge a 1 in. overhang was allowed. The thickness of the edge strips was 2 in. in all cases, and the nominal thickness of the tested area was either 2 in., 1½ in., or 1 in. The thicknesses of the tested areas of the unreinforced slabs were checked after the slabs had been tested. Micrometer readings were taken at 16 stations along the yield lines and the mean thickness of each slab, except slab E8, showed variations of up to ± 0.05 in. from the expected thickness. For the 2 in. slabs this meant a maximum deviation of only ± 2½% from the nominal thickness and the nominal thickness was used in the calculations. For the * All the slabs of series E were cured for 7 days, except slab E1 which was cured for 10 days.
thinner slabs and slab E8, however, the variation was more appreciable and the exact thickness measured was used.

The details of the reinforcement and the edge conditions of each slab were as follows:

**Slab Series A: Al to A4**

The four slabs of this series were tested with all edges fully restrained. The slabs were reinforced as if designed by Johansen's yield-line theory for the minimum weight of reinforcement for various ultimate loads. (see Appendix A). The minimum steel contents allowed by the British code of practice°F was placed in the bottom of the slabs in the direction of the long span. The amount of steel placed in the bottom of the slabs in the direction of the short span varied upwards from the minimum amount allowable. The cross-sectional area of top steel, per unit width, was twice that of the bottom steel. The top and bottom bars in the direction of the short span were placed with 1⁄4in. cover. The long span bars were placed with greater cover and tied to the short span bars. For all slabs the spacing of the bars in both directions was 1⁄4in. in the bottom and 2in. in the top. The top and bottom bars in the same direction were of the same diameter, but the diameters were not necessarily the same in both directions as is shown by Table B.2. The
bottom steel was placed over the whole of the tested areas of the slabs, but the top steel was placed only around the edges of the tested area of each slab. The length of the top steel was calculated by Johansen's theory as being just sufficient to ensure that the hogging moment yield lines formed along the lines of the supports. As is shown in Appendix A, this meant that each top bar extended into 0.21 of the span, and a further 12 diameters of length was provided for anchorage before stopping off. The top and bottom steel was continued to the outside edges of the edge strips. The layout of the reinforcement of slab A3 is shown in Fig.B.2. Table B.3 shows the dimensions and the properties of the concrete and steel of the slabs.

**Slab Series B and C: B1 to B4, C1 to C4.**

The four slabs of series B were tested with one short edge simply supported on rollers and the four slabs of series C were tested with one long edge simply supported on rollers, the remaining edges of both series being fully restrained. The steel contents of slabs B1 and C1 were identical to that of slab A1, except that there was no top steel at the simply supported edge and that the length of the other top steel was such as to suit their particular boundary conditions. Similarly the steel contents of slabs A2, B2 and C2 were identical.
as were slabs A3, B3 and C3, and slabs A4, B4 and C4, with the exception that top steel was omitted at the simply supported edges and the length of the other top steel was such as to suit the particular slab. As is shown in Appendix A, the top steel at the fixed edges was required to extend into 0.22 and 0.24 of the spans for the slabs of series B and C, respectively. A further 12 diameters was provided for anchorage before stopping off. Figure B.3 shows the layout of the reinforcement of slab C4.

Table B.3 shows the dimensions and the properties of the concrete and steel of the slabs. The slight differences in yield forces between the slabs with identical reinforcement contents comes about from the slight differences in the yield stresses.

**Slab Series D and E: D1 to D5, E1 to E8.**

The 13 slabs of these series were tested with all edges fully restrained. The slabs were unreinforced. Tables B.4 and B.5 show the dimensions and the properties of the concrete.
Table B.3: Slab Series A, B and C: Dimensions and Properties of Concrete and Steel.

<table>
<thead>
<tr>
<th>Slab</th>
<th>Age at Test. Days</th>
<th>Age at Test.</th>
<th>$T_x = C_{sx}$</th>
<th>$T_y = C_{sy}$</th>
<th>$d_x = d'_y$</th>
<th>$d_y = d'_y$</th>
<th>$d'_{x}$</th>
<th>$d'_{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>110</td>
<td>5970</td>
<td>163</td>
<td>163</td>
<td>1.69</td>
<td>1.56</td>
<td>0.31</td>
<td>0.44</td>
</tr>
<tr>
<td>A2</td>
<td>114</td>
<td>5350</td>
<td>280</td>
<td>153</td>
<td>1.66</td>
<td>1.50</td>
<td>0.34</td>
<td>0.50</td>
</tr>
<tr>
<td>A3</td>
<td>117</td>
<td>6250</td>
<td>603</td>
<td>163</td>
<td>1.63</td>
<td>1.44</td>
<td>0.37</td>
<td>0.56</td>
</tr>
<tr>
<td>A4</td>
<td>110</td>
<td>5020</td>
<td>870</td>
<td>163</td>
<td>1.59</td>
<td>1.37</td>
<td>0.41</td>
<td>0.63</td>
</tr>
<tr>
<td>B1</td>
<td>113</td>
<td>4450</td>
<td>160</td>
<td>160</td>
<td>1.69</td>
<td>1.56</td>
<td>0.31</td>
<td>0.44</td>
</tr>
<tr>
<td>B2</td>
<td>114</td>
<td>4520</td>
<td>313</td>
<td>153</td>
<td>1.66</td>
<td>1.50</td>
<td>0.34</td>
<td>0.50</td>
</tr>
<tr>
<td>B3</td>
<td>119</td>
<td>4680</td>
<td>603</td>
<td>153</td>
<td>1.63</td>
<td>1.44</td>
<td>0.37</td>
<td>0.56</td>
</tr>
<tr>
<td>B4</td>
<td>117</td>
<td>4720</td>
<td>845</td>
<td>141</td>
<td>1.59</td>
<td>1.37</td>
<td>0.41</td>
<td>0.63</td>
</tr>
<tr>
<td>C1</td>
<td>92</td>
<td>6250</td>
<td>152</td>
<td>150</td>
<td>1.69</td>
<td>1.56</td>
<td>0.31</td>
<td>0.44</td>
</tr>
<tr>
<td>C2</td>
<td>93</td>
<td>5480</td>
<td>313</td>
<td>150</td>
<td>1.66</td>
<td>1.50</td>
<td>0.34</td>
<td>0.50</td>
</tr>
<tr>
<td>C3</td>
<td>78</td>
<td>5080</td>
<td>593</td>
<td>150</td>
<td>1.63</td>
<td>1.44</td>
<td>0.37</td>
<td>0.56</td>
</tr>
<tr>
<td>C4</td>
<td>77</td>
<td>6360</td>
<td>878</td>
<td>150</td>
<td>1.59</td>
<td>0.37</td>
<td>0.41</td>
<td>0.63</td>
</tr>
</tbody>
</table>

Note: 1. For all slabs: $L_y = 60$ in., $L_x = 40$ in., $d = 2.0$ in.
2. The edges of slab series A were fully restrained.
   Slab series B had a short edge simply supported and slab series C had a long edge simply supported, with all other edges fully restrained.
3. It is assumed that $C_{sx} = C_{sy} = 0$, since top steel was only placed around the edges. It is also assumed that the remaining compression steel was yielding.
4. At the simply supported edges, $T'_y = 0$ for series B and $T'_x = 0$ for series C.
### Table B.4: Slab Series D: Dimensions and Properties of Concrete

<table>
<thead>
<tr>
<th>Slab</th>
<th>Age at Test. Days</th>
<th>u (lb/sq.in.)</th>
<th>d (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>76</td>
<td>6280</td>
<td>2.0</td>
</tr>
<tr>
<td>D2</td>
<td>72</td>
<td>6200</td>
<td>1.50</td>
</tr>
<tr>
<td>D3</td>
<td>60</td>
<td>6430</td>
<td>0.98</td>
</tr>
<tr>
<td>D4</td>
<td>52</td>
<td>5550</td>
<td>1.01</td>
</tr>
<tr>
<td>D5</td>
<td>14</td>
<td>4440</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Note: 1. For all slabs: $L_y = 60$ in., $L_x = 40$ in.
2. All edges were fully restrained.
3. The slabs were unreinforced.

### Table B.5: Slab Series E: Dimensions and Properties of Concrete

<table>
<thead>
<tr>
<th>Slab</th>
<th>d (in.)</th>
<th>At First Load</th>
<th>At End of Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Age Days</td>
<td>u (lb/sq.in.)</td>
</tr>
<tr>
<td>E1</td>
<td>0.97</td>
<td>14</td>
<td>4790</td>
</tr>
<tr>
<td>E2</td>
<td>0.95</td>
<td>14</td>
<td>3620</td>
</tr>
<tr>
<td>E3</td>
<td>0.98</td>
<td>14</td>
<td>3720</td>
</tr>
<tr>
<td>E4</td>
<td>0.95</td>
<td>151</td>
<td>5120</td>
</tr>
<tr>
<td>E5</td>
<td>0.98</td>
<td>151</td>
<td>6340</td>
</tr>
<tr>
<td>E6</td>
<td>1.48</td>
<td>14</td>
<td>4810</td>
</tr>
<tr>
<td>E7</td>
<td>1.54</td>
<td>14</td>
<td>4990</td>
</tr>
<tr>
<td>E8</td>
<td>2.07</td>
<td>14</td>
<td>4260</td>
</tr>
</tbody>
</table>

Note: 1. For all slabs: $L_y = 60$ in., $L_x = 40$ in.
2. All edges were fully restrained.
3. The slabs were unreinforced.
B.2 The Control Specimens

B.2.1 Slabs of Series A, B, C and D: Short-term Load Tests.

Four 6in. cubes were cast with each slab, one from each batch of concrete mixed. The mean compressive strength of the four cubes and the age of the concrete when the slabs were tested to failure are given in Tables B.3 and B.4. All specimens were cured with and in the same fashion as the slabs.

B.2.2 Slabs of Series E: Sustained Loading Tests

All specimens were cured with and in the same fashion as the slabs.*

(a) Compressive Strength Specimens

With each slab eight 6in. concrete cubes were cast, two from each batch of concrete mixed. Four cubes, one from each batch, were tested at the stage when the sustained loading was first applied to the slab, and four cubes were tested at the stage when the slab was loaded to failure. The mean cube strengths at each stage are shown in Table B.5.

(b) Young’s Modulus and Creep Specimens

The specimens were cylinders, 2in. diameter by 10\(\frac{5}{8}\) in. long. A total of 4 representative pairs of specimens were cast from the same concrete as the slabs.

* The Young’s modulus, creep and shrinkage specimens were cured for 7 days and hence were representative of all slabs except E1.
(c) **Shrinkage Specimens**

The specimens were either cylinders, 2in. diameter by 10\(\frac{1}{2}\)in. long, or small slabs of 1\(\frac{1}{2}\)in. thickness. A total of 5 representative pairs of shrinkage specimens were cast from the same concrete as the slabs.

B.3. **The Test Frame, Loading and Instrumentation**

B.3.1 **The Test Frame**

The base of the test frame is shown in Fig.B.4. It is similar to the type used by Powell\(^5\). The frame was fabricated mainly from 1\(\frac{1}{2}\)in. thick mild steel plate and formed an extremely stiff rectangular ring. The sides of the frame were of two-cell box section. The frame was designed to be capable of testing slabs of up to 3in. thickness and to be stiff enough to make the elastic deformations of the sides extremely small. The frame was bolted down onto three 18in. x 12in. x 122lb. B.F.B.'s as shown in Fig.B.5. The methods of supporting the slab edges are also shown in Fig.B.5. The rotation and vertical translation of edges which were to be fully restrained were prevented by clamping the edge strips down under 12in. x 4in. channels using 1\(\frac{1}{2}\)in. diameter bright steel studs as shown in Fig.B.5a. High strength grout was placed
FIG. B.4 DETAILS OF TEST FRAME.
1" dia. high tensile screws at 6" centres.

1" dia. bright steel studs at 6" centres.

12"x4" channel.

Steel plate.

1" dia. roller.

Rubber bag.

Water.

18"x12" B.F.B.

FIG. B.5 DETAILS OF EDGE SUPPORTS.

a. Fully Restrained Edge.

b. Simply Supported Edge.
between the channels, the edge strips and the test frame, to obtain even bearing. Horizontal movement of the slab edges was restrained by the bond due to the grout, by friction induced by pulling the clamping channels down hard onto the edge strips, by bearing against the studs (grout was forced into the annular holes between studs and edge strips), and by high tensile steel bolts bearing horizontally against steel plates at the edges of the slabs. The horizontal restraint bolts were only done up finger tight to ensure that the slabs were not "prestressed" before testing. Edges which were to be simply supported on rollers were supported as shown in Fig.B.5b. To obtain even bearing, grout of the required thickness was placed between the slab and the bearing plates of the rollers.

**B.3.2 Method of Loading**

The uniformly distributed loading was applied upwards using a rubber bag filled with water at the required pressure. The bag was placed under the slab and was surrounded by the slab, the sides of the frame and a sheet of 5/8in. thick mild steel plate as shown in Fig.B.5. The bag was provided by I.C.I.Ltd. and was made from their MM339 material. Water pressure in preference to air pressure was used for loading since a near incompressible fluid was
desirable for safety in the event of sudden fracture of the slab. The use of water rather than air also allowed the descending branch of the load-deflection curve after ultimate flexural load to be followed. The bag pressure was measured by Bourdon pressure gauges of ranges 0 to 60 lb./sq.in. and 0 to 30 lb./sq.in., or a mercury U tube manometer of 0 to 16 lb./sq.in. range. Both Bourdon pressure gauges were calibrated at frequent intervals. To eliminate any possible effects of friction, two layers of P.V.C. sheeting were placed between the bag and the loaded face of the slab, and French chalk was sprinkled between all interfaces.

Although the bag was slightly larger than the cavity it had to fill and was capable of loading the slab up to a central deflection of approximately 6in., there was a narrow band around the edges of the tested areas of the slabs where the bag was not in contact with the slabs. To compare the pressure applied to the slab with the U tube manometer reading a loading test was conducted on one occasion. Proving rings bearing against the top of the slab were used to determine the total upward force applied to the slab and it was found that the pressure applied to
the slab as read by the U tube manometer was 8% greater than the apparent pressure measured by the proving rings (calculated by dividing the total reaction measured by the proving rings by the area of the slab between supports). This could be accounted for by a narrow band 0.85 in. wide around the tested area of the slab where the bag was not in contact with the slab. In the ultimate strength calculations for the slab no allowance has been made for zero loading on this narrow edge band since it can be shown that the effect of it is to increase the ultimate strength of the slab by only 0.4%.

The details of the method of applying the load were as follows:

(a) Short-Term Loading Tests

Details of the loading arrangements are shown in Fig.B.6a. Care was taken to eliminate air from the system. Approximately 90 lb./sq.in. pressure was available from the mains water supply. The slabs were loaded using a fine adjustment inlet valve to introduce water into the bag until the required pressure was reached. Close control over the water pressure in the bag was obtained. Use of the Bourdon gauges and a U tube manometer with different maximum capacities meant that accurate readings of pressure were obtained over the whole of the loading range.
a. Short-Term Loading Tests.

b. Sustained Loading Tests.

FIG. B.6 METHODS OF LOADING SLABS.
(b) Sustained Loading Tests

Details of the loading arrangements are shown in Fig. B.6b. Mains water supply was used to load the slabs up to the load to be sustained. The loading was then held by an over-head tank which provided a constant head of water to the loading bag. Use of a constant head tank meant that the mains supply could be disconnected once the loading was applied. Due to evaporation and the slight increase in volume of the bag resulting from the increase in deflection of the slab during the period of sustained loading, the tank had to be topped up every few days. The fluctuation in load resulting from these small changes in head was never more than 0.5% of the required sustained load, however.

B.3.3 Time and Rate of Loading

The ages of the slabs when loaded are shown in Tables B.3, B.4 and B.5. The loading was applied to each slab in increments in order to measure deflections of the slab surface and the extent of cracking at various stages. In the short-term loading tests the time taken to load each slab to failure varied between 1½ to 2½ hours. In the sustained load tests the time taken to load each slab up to the load to be sustained varied between 15 and 50 minutes. The sustained load was left on the slabs for 42 days,
except for E1 which failed after less than 24 hours of sustained load. The loading of the slabs to failure after the period of sustained loading took 15 to 55 minutes.

B.3.4 Instrumentation.

Vertical deflections were measured at various points on the surfaces of the slabs using Mercer dial gauges. Some of the readings taken have been quoted in previous chapters. Measurements of the lateral movement of the outside edges of the slabs were also taken and have been referred to previously where necessary. Some strain gauge readings were also taken.

B.4 The Loading and Instrumentation of the Young's Modulus, Creep and Shrinkage Specimens.

Details of the sizes of the test specimens associated with the slabs of series E are given in Section B.2.2. The strains on the specimens were measured using a Demec strain gauge with an 8in. gauge length.

(a) Young's Modulus Specimens

Longitudinal strains were measured on axially loaded cylinders to determine Young's modulus at the stage of first applying the load to the slabs and at the stage of loading the slabs to failure. The readings were taken as quickly as possible to avoid creep. Young's modulus was
taken as the tangent to the stress-strain curve at zero stress. Table B.6 shows the values obtained and the cube strengths of the concrete at the time. The creep specimens were used for these tests, the stress-strain curves being obtained when the specimens were being loaded up to the sustained stress, and after the period of sustained stress.

**Table B.6: Young's Modulus of Concrete Specimens**

<table>
<thead>
<tr>
<th>Specimen Mark</th>
<th>First Test</th>
<th></th>
<th>Second Test</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$u_{lb/sq.in.}$</td>
<td>$E_c \times 10^6_{lb/sq.in.*}$</td>
<td>$u_{lb/sq.in.}$</td>
</tr>
<tr>
<td>IA</td>
<td>5680</td>
<td>4.1</td>
<td>6730</td>
<td>4.5</td>
</tr>
<tr>
<td>IB</td>
<td>5680</td>
<td>4.3</td>
<td>6730</td>
<td>4.6</td>
</tr>
<tr>
<td>IIA</td>
<td>4280</td>
<td>3.6</td>
<td>5760</td>
<td>3.8</td>
</tr>
<tr>
<td>IIB</td>
<td>4280</td>
<td>3.3</td>
<td>5760</td>
<td>3.5</td>
</tr>
<tr>
<td>E8A</td>
<td>4260</td>
<td>3.6</td>
<td>5140</td>
<td>4.4</td>
</tr>
<tr>
<td>E8B</td>
<td>4260</td>
<td>3.3</td>
<td>5140</td>
<td>4.3</td>
</tr>
<tr>
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<td>4.3</td>
<td>6450</td>
<td>4.5</td>
</tr>
<tr>
<td>E5B</td>
<td>6340</td>
<td>4.1</td>
<td>6450</td>
<td>4.6</td>
</tr>
</tbody>
</table>

* Mean of two readings on each specimen.

(b) *Creep Specimens*

Longitudinal creep strains were measured on axially loaded cylinders under a constant sustained stress for the period of time which the slabs were under sustained loading.
Hence the specimens associated with slabs E2, E3, E6, E7 and E8 were loaded when the concrete was 14 days old and left loaded for 42 days, and the specimens associated with slabs E4 and E5 were loaded when the concrete was 151 days old and left loaded for 42 days. The longitudinal strains were measured on opposite sides of a diameter of each cylinder. Shrinkage strains which occurred during the period of sustained loading were subtracted from the measured strains to give the true creep strains.

Two frames of the type shown in Fig.B.7 were used to apply the creep loads. These frames were very similar to the type described by Neville\textsuperscript{14}. In each frame the load was held constant by a spiral spring. The load applied to each specimen was determined from the strains on the tie rod of the frame and checked by the deflection of the spiral spring. The creep frames were calibrated by proving rings before use.

Figure B.8 shows the measured creep strains plotted against time of loading. Table B.7 shows the cube strengths, the sustained stresses of the specimens and the final creep strains. The creep frames were placed beside the slabs under load and hence the conditions of temperature and humidity were the same.
FIG. B.7 LOADING FRAME FOR CREEP SPECIMENS.

Loading handle.
Ball race.
Spherical washer.

Spiral spring.
Stiffness: \( \frac{2}{3} \) ton/in.

Knife edge.

4" x 3" R.S.J.

Demec strain gauge points.

2" dia. concrete specimen.

0.5" dia. bright steel rod.

Steel cap

Spherical washer.

4" x 3" R.S.J.

23.75"

21.25"

4" x 2" Channel.
Specimens E5A and E5B were loaded at age 151 days. The other specimens were loaded at age 14 days.

FIG. B.8 SPECIFIC CREEP STRAINS OF CONCRETE SPECIMENS DURING LOADING PERIODS.
Table B.7: Creep Strains From Specimens

<table>
<thead>
<tr>
<th>Specimen Mark</th>
<th>u at 1st load. lb/sq.in.</th>
<th>Sustained stress u at 1st load</th>
<th>Age of Specimen. Days</th>
<th>Creep Strain During Period of Sustained loading. ( x 10^{-6} )</th>
<th>( \varepsilon_c ) per lb/sq.in. stress ( x 10^{-6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>IA</td>
<td>5680</td>
<td>0.358</td>
<td>14</td>
<td>56</td>
<td>1052</td>
</tr>
<tr>
<td>IB</td>
<td>5680</td>
<td>0.347</td>
<td>14</td>
<td>56</td>
<td>882</td>
</tr>
<tr>
<td>II A</td>
<td>4260</td>
<td>0.278</td>
<td>14</td>
<td>56</td>
<td>843</td>
</tr>
<tr>
<td>IIB</td>
<td>4260</td>
<td>0.264</td>
<td>14</td>
<td>56</td>
<td>813</td>
</tr>
<tr>
<td>E8 A</td>
<td>4260</td>
<td>0.345</td>
<td>14</td>
<td>56</td>
<td>862</td>
</tr>
<tr>
<td>E8 B</td>
<td>4260</td>
<td>0.331</td>
<td>14</td>
<td>56</td>
<td>947</td>
</tr>
<tr>
<td>E5 A</td>
<td>6340</td>
<td>0.399</td>
<td>151</td>
<td>193</td>
<td>498</td>
</tr>
<tr>
<td>E5 B</td>
<td>6340</td>
<td>0.360</td>
<td>151</td>
<td>193</td>
<td>473</td>
</tr>
</tbody>
</table>

* Mean of two readings on each specimen.
(c) **Shrinkage Strains**

Shrinkage strains were measured on either unloaded cylinders or small slab specimens. The shrinkage occurring in the time between clamping the slabs into the test frame and loading the slabs to failure was measured. For the specimens associated with slabs E2, E3, E6, E7 and E8 this meant commencing readings when the concrete was 10 days old and continuing readings for 46 days. For the specimens associated with slabs E4 and E5 readings were commenced when the concrete was 147 days old and continued for 46 days. The shrinkage specimens were placed beside the slabs under load and hence the temperature and humidity conditions were the same. Figure B.9 shows the shrinkage strains plotted against time and Table B.8 shows the final strains of all specimens.
FIG. B.9 SHRINKAGE STRAINS OF CONCRETE SPECIMENS DURING TIME SLABS WERE IN TEST FRAME.
**Table B.8: Shrinkage Strains From Specimens**

<table>
<thead>
<tr>
<th>Specimen Mark</th>
<th>Age of Specimen Days.</th>
<th>Shrinkage Strain $\varepsilon_c \times 10^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>At 1st Reading</td>
<td>At last Reading</td>
</tr>
<tr>
<td>IA</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>IB</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>IIA</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>IIB</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>E7A</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>E7B</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>E8A</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>E8B</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>E5A</td>
<td>147</td>
<td>193</td>
</tr>
<tr>
<td>E5B</td>
<td>147</td>
<td>193</td>
</tr>
</tbody>
</table>

* Mean of two readings on each specimen
** Mean of three readings on each specimen.

The cube strengths of specimens were as in Table B.7 for the appropriate mark.
Appendix C: Slab Series F and G: Details of Slabs and Testing Arrangements.

C.1. The Slabs

(a) Properties of the Mortar Mix and the Reinforcement

The slabs were made from a sand-cement mortar mix. The grading of the sand was as follows:

<table>
<thead>
<tr>
<th>B.S. Sieve Size</th>
<th>No.14</th>
<th>No.25</th>
<th>No.52</th>
<th>No.100</th>
</tr>
</thead>
<tbody>
<tr>
<td>% Passing, by weight</td>
<td>100.0</td>
<td>74.9</td>
<td>8.1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Rapid hardening Portland cement was used. The aggregate/cement ratio was 4.0 and the water/cement ratio was in the range 0.66 to 0.73, both by weight. The compressive strength and the age of the mortar when each slab was tested are shown in Table C.1. The compressive strength was the mean result obtained from three or four 4 in. cubes.

The reinforcement was of bright steel bar, either 1/8 in. square for series F or 1/4 in. square for series G. Tension tests showed that the modulus of elasticity of this steel was $30 \times 10^6$ lb./sq.in. and that the limit of proportionality was not less than 45,000 lb./sq.in. The details of yield stresses and ultimate tensile strengths will be omitted since this steel was not stressed beyond the limit of proportionality when in the slabs.
(b) The Mould and Casting

The mould was formed from a plywood sheet which was stiffened on the under side by timber and mounted on rubber pads. The sides of the mould were lengths of bright steel bar. The reinforcement was held at mid-depth of the slabs by wire chairs. The mortar was compacted by vibration and the slabs were cured for periods of 4 to 7 days by placing damp hessian over the mortar surface.

(c) Slab Dimensions and Reinforcement Contents

The slabs were made to simulate nine panel floors with panels arranged three by three. The interior panel of each slab was 12in. square and the surrounding edge panels were of the dimensions indicated in Figs. 5.9 and 5.10. Loading was applied only to the interior panel and after testing each slab the thickness of the interior panel was measured by micrometer readings taken at sixteen stations along the yield lines. The mean thickness of the interior panel of each slab is shown in Table C.1.

The interior panel of each slab was unreinforced. The edge panels were reinforced as shown in Figs. 5.9 and 5.10. The right angle bends of the "square ring" reinforcement were of 1in. radius and the hooks at the ends of the straight bars were of ¹⁄₂in. radius. The square rings were continuous,
the bars having been joined by welding or some other fusion process along laps at least 2in. long. Tests confirmed that the joint developed the strength of the bar.

Table C.1. Slab Series F and G: Thickness and Strength of Mortar.

<table>
<thead>
<tr>
<th>Slab</th>
<th>Age. Days</th>
<th>F1</th>
<th>F2</th>
<th>F3</th>
<th>F4</th>
<th>F5</th>
<th>F6</th>
<th>F7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>12</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>u lb/sq.in.</td>
<td></td>
<td>4030</td>
<td>3500</td>
<td>3760</td>
<td>4350</td>
<td>3240</td>
<td>3680</td>
<td>2770</td>
</tr>
<tr>
<td>d in.</td>
<td></td>
<td>0.354</td>
<td>0.357</td>
<td>0.369</td>
<td>0.379</td>
<td>0.388</td>
<td>0.385</td>
<td>0.373</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Slab</th>
<th>Age. Days</th>
<th>F8</th>
<th>F9</th>
<th>F10</th>
<th>F11</th>
<th>F12</th>
<th>F13</th>
<th>F14</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>7</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>u lb/sq.in.</td>
<td></td>
<td>3030</td>
<td>3400</td>
<td>3170</td>
<td>2750</td>
<td>2830</td>
<td>3870</td>
<td>3050</td>
</tr>
<tr>
<td>d in.</td>
<td></td>
<td>0.371</td>
<td>0.378</td>
<td>0.390</td>
<td>0.401</td>
<td>0.411</td>
<td>0.390</td>
<td>0.398</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Slab</th>
<th>Age. Days</th>
<th>G1</th>
<th>G2</th>
<th>G3</th>
<th>G4</th>
<th>G5</th>
<th>G6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>7</td>
<td>14</td>
<td>7</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>u lb/sq.in.</td>
<td></td>
<td>2490</td>
<td>2300</td>
<td>3460</td>
<td>2450</td>
<td>3250</td>
<td>2670</td>
</tr>
<tr>
<td>d in.</td>
<td></td>
<td>0.693</td>
<td>0.690</td>
<td>0.667</td>
<td>0.674</td>
<td>0.663</td>
<td>0.640</td>
</tr>
</tbody>
</table>

C.2. The Test Frame and Loading

The base and top of the test frame are shown in Figs. C.1 a and b. The base consisted of a square sheet of 0.25in. thick mild steel plate which was stiffened on the underside by 4in. x 2in. channels. In the centre of the plate a 12in. square hole was cut and a sheet of 12in. square plate was welded to the bottom of the channels thus forming a cavity to hold the loading bag. The 12in. square interior panel of the slab was loaded upwards. The slab was supported by 0.25in.
FIG. C1 TEST FRAME: SLAB SERIES F AND G.

a. Base of Test Frame.

b. Top of Test Frame.

c. Section Across Test Frame With Slab in Position.
diameter rollers placed around the outside edges of the floor and around the edges of the interior panel, as shown in Fig.C.1c and Fig.5.8. The rollers around the edges of the interior panel were held down by a top frame formed from 4in. x 2in. channel. The rollers allowed the slab to move laterally but prevented vertical translation along the lines of the supports. Fig.C.2 shows the test frame before and after setting up a slab for testing.

The loading bag was made from "Li-Lo" material. P.V.C. sheets were placed between the bag and the slab to eliminate the effects of friction. The bag was filled with water from the mains and a fine adjustment inlet valve used to obtain the required bag pressure. The water pressure in the loading bag was measured using mercury U tube manometer or a Bourdon pressure gauge. The slabs were loaded to failure over a period of approximately half an hour, the load being applied in increments in order to measure deflections and to observe cracking.
a. Without Slab.

b. With Slab.

FIG. C.2 TEST FRAME: SLAB SERIES F AND G.
REFERENCES

1. British Standard Code of Practice CP 114 (1957):
   "The Structural Use of Reinforced Concrete in Buildings".


10. E. Hognestad, N.W. Hanson & D. McHenery: "Concrete Stress Distribution in Ultimate Strength Design". Journal of American Concrete Institute, December 1955.


13. Comité Européen du Béton (European Concrete Committee), Committee 1: "Practical Recommendations for Use by Designers". March 1961.


Appendix A: Design of Reinforced Concrete Slabs By Yield-Line Theory

DESIGN OF REINFORCED CONCRETE SLABS BY YIELD-LINE THEORY

R. PARK*
M.E. GRAD. N.Z.I.E.

The yield-line method for the ultimate strength design of two-way reinforced concrete slabs is outlined. The theory takes into account the redistribution of bending moment that occurs after yielding of steel has commenced, and considers collapse of the slab to occur when sufficient lines of yielding have developed to divide the slab up into segments connected by plastic hinges which can form a mechanism. Basic principles of the theory are given and the methods of analysis explained. Expressions suitable for the design of uniformly loaded two-way rectangular slabs with any combination of fixed or simply supported edges are given and the point of curtailment of reinforcement for hogging bending moment is considered. The manner in which the reinforcement should be distributed to give minimum weight of steel for a given strength of slab is investigated. Finally an example design is worked and the results compared with conventional design.

1. INTRODUCTION

ULTIMATE strength methods of design are rapidly becoming more widely accepted by codes of practice and more commonly used by designers. This trend is shown by the British Standard C.P.114 (1957) which allows design of uniformly loaded two-way slabs by a load-factor method which gives due regard to the redistribution of bending moments which occur before the ultimate load is reached. The deflection at working load is controlled by specifying maximum allowable values of span/depth ratio. The British code is not alone in allowing load-factor design of reinforced concrete slabs. The Danish code has for some time allowed this form of design, although the present American Concrete Institute Building Code A.C.I.318-56 does not. The load-factor method of design is becoming popular since it allows the designer more freedom when positioning reinforcement in the slab and the method has the advantage of simplicity, even when analysing slabs with unusual shape and boundary conditions. The method can also result in economy of steel.

The method of limit analysis of slabs termed "yield-line theory" was initiated by Ingerslev (1) and was greatly extended and advanced by Johansen, e.g. (2), (3). The greater part of this early work, however, is published in Danish and probably the four most comprehensive English references are (4), (5), (6) and (7). Yield-line theory has had sufficient experimental verification to enable it to be used with confidence, as is shown by its adoption by B.S.C.P. 114 (1957).

In this paper an effort has been made to present yield-line theory results in a form suitable for the design office.

2. BASIC PRINCIPLES OF YIELD LINE THEORY

2.1. Reinforcement

The theory is applicable to concrete slabs of constant thickness which are reinforced uniformly in two directions. The cross-sectional area of reinforcement per unit width, however, may be different in the two directions. For such slabs the ultimate moment of resistance per unit width will have a constant value along any straight line which lies within the plane of the slab.

In the design of reinforced concrete slabs the need to limit deflections will generally lead to relatively low percentages of steel, rather less than 1.0% being almost always the case, and hence if a typical reinforced concrete slab is loaded to its ultimate strength, failure will be initiated by yielding of the tension steel rather than by crushing of the concrete.

2.2. Conditions at Yielding and Collapse

When a reinforced concrete slab is progressively loaded until the tension steel at the section of maximum bending moment has begun to yield, the slab undergoes a large change in curvature along the line of yielding with little increase in the resisting moment of the section. The curvature at which crushing of the concrete finally takes place may be some 10-15 times the curvature at which yielding of steel commenced and the resisting moment-curvature curve has the shape shown in Fig. 1. Thus, at the commencement of yielding of steel, the slab has not reached its maximum load-carrying capacity since the rotation available along the line of yielding at practically constant bending moment allows a significant redistribution of bending moment.

As the load on the slab is increased beyond that which caused the initial yielding, the lines of yielding extend and additional load can be carried until the yield lines have developed in sufficient numbers to divide the slab up into segments which can form a mechanism. Al-
though yielding first occurs at the section of maximum bending moment given by elastic theory, the positions of the lines of yielding developed by further loading are governed by the arrangement of reinforcement, the boundary conditions and the type of loading. Fig. 2 shows the development of a typical yield line pattern. It is to be noted that the yield line is an idealization since in fact the cracks form in bands, as in Fig. 3. For the purpose of analysis, however, each band is represented by a single yield line at the centre of the band and all rotation is considered to occur along that line.

![Fig. 3.](image)

2.3. Yield Lines as Axes of Rotation

As the plastic deformations along yield lines are much larger than the elastic deformations of the slab portions between the yield lines when the collapse mechanism has developed, it is reasonable to assume that the individual slab segments between the yield lines are plane. To act as plastic hinges of a collapse mechanism, yield lines must be straight lines forming axes of rotation for the movements of the segments of the collapse mechanism. Examination of the geometry of deformations gives two basic rules for determining the pattern of yield lines:

(i) The supports of the slab will act as axes of rotation. If an edge is fixed a yield line may form along the support.

(ii) For compatibility of deformations, a yield line must pass through the intersection of the axes of rotation of the adjacent slab segments.

Fig. 4 shows examples of typical yield-line patterns developed in slabs with various boundary conditions. The angles between yield lines are found by methods outlined later in the paper.

![Fig. 4.](image)

2.4. Yield Moments

The yield moment of a section, defined as the ultimate resisting moment of that section neglecting strain hardening of steel, may be found by any plastic theory, e.g., Whitney’s theory or that given in B.S.C.P. 114. For a slab with reinforcement in the x and y directions, in the general case the yield moments per unit width in the x and y directions will be unequal since the areas of steel per unit width and the effective depths of steel will be different in those directions. For this case it is sometimes necessary to determine the yield moment per unit width along a line at angle other than 90° to the x and y axes. If $M_x$ and $M_y$ are the yield moments per unit width in the x and y directions respectively, then the yield moment per unit width in a direction at angle $\alpha$ to the x axis is:

![Diagram of yield moment](image)

**NOTATION**

- $x, y$: Rectangular axes in the plane of the slab.
- $L_x, L_y$: Length of slab in x and y directions respectively with $L_y > L_x$.
- $L$: Length of a yield line.
- $L_1, L_2, \ldots$: Dimensions which define the positions of yield lines.
- $M_x, M_y$: Sagging yield moments, per unit width, acting in x and y directions respectively.
- $M'_x, M'_y$: Hogging yield moments, per unit width, acting in x direction at opposite edges of slab.
- $M''_x, M''_y$: Hogging yield moments, per unit width, acting in y direction at opposite edges of slab.
- $\mu = \frac{M_x}{M_y} = \frac{M'_x}{M'_y}$
- $\eta = \frac{M''_x}{M''_y}$
- $M_x$: Yield moment per unit width acting in a direction at angle $\alpha$ to the x axis.
- $\theta_x$: Virtual rotation of a slab segment about a yield line which is inclined at angle $\alpha$ to the y axis.
- $\theta_x, \theta_y$: Components of $\theta_x$ in x and y directions respectively.
- $\delta(x, y)$: Virtual displacement of point x, y of slab.
- $W_u$: Collapse load per unit area.
- $W_u'$: Total collapse load.
- $\lambda$: Coefficient defining length of hogging moment steel as a proportion of the span.
- $A_{sx}, A_{sy}$: Cross-sectional area of x and y direction tension steel, respectively, for sagging moment, per unit width.
- $A'_{sx}, A'_{sy}$: Cross-sectional area of x and y direction tension steel, respectively, for hogging moment, per unit width.
- $P_{st}$: Working stress for steel in tension.
- $P_{ceb}$: Working stress for concrete in compression due to bending.
- $d_t$: Effective depth of tension steel.
- $V$: Volume of steel in slab.
Fig. 5 shows the directions of reinforcement and yield moments. The x and y directions (the directions of the reinforcement) are taken to be the directions of principal moments.

\[
M_x = M_x \cos^2\alpha + M_y \sin^2\alpha 
\]

(1)

2.5. Intersection of Yield Lines and Edges

Twisting moments and shear forces can occur along yield lines as well as bending moments and it is convenient to replace the action of shear forces by statically equivalent concentrated forces (termed nodal forces) acting at the ends of the yield lines. When, however, yield lines intersect at a point and are either all hogging moment or all sagging moment lines, each nodal force at the point is zero. When a yield line enters a free edge at an angle of other than 90° to the edge (see Fig. 6), and the reinforcement runs parallel and perpendicular to the edge, the nodal forces at the edge are:

\[
S = M \cot \phi \quad \ldots \quad \ldots \quad \ldots \quad (2)
\]

Here \( \phi \) is the acute angle and \( M \) the yield moment per unit width in the direction perpendicular to the edge.

For a sagging moment yield line \( S \) acts downwards in the acute angle corner. In practice the yield line curves close to the edge to enter at 90°, but a straight yield line is assumed.

The virtual work done by the internal actions of the slab will be due only to the yield moments at yield lines, since the virtual work done by the twisting moments and the shear forces at yield lines is zero when summed over the whole slab. The virtual work done by yield moment per unit width \( M_x \) which is acting in direction at angle \( \alpha \) to the \( x \) axis along the yield line of length \( L \) (at angle \( \alpha \) to \( y \) axis) at which the virtual rotation is \( \theta_x \) is:

\[ -\int_0^L M_x \theta_x \, dx \]

The work done is negative since the yield moment will be acting in the opposite direction to the virtual rotation. Substituting for \( M_x \) from equation (1), the virtual work done along the yield line is:

\[ -\int_0^L (M_x \cos^2\alpha + M_y \sin^2\alpha) \theta_x \, dx \]

(3)

In which the integrations are made along each yield line and the contributions from all yield lines are summed.

The virtual work done by the load per unit area at collapse, \( w_x \), when the slab undergoes virtual displacement \( \delta(x, y) \) is:

\[ w_x = \int \delta(x, y) \, dx \]

3. ANALYSIS BY YIELD-LINE THEORY

3.1. Method of Analysis:

The collapse load is defined as the minimum load required to reduce the slab into a mechanism. The collapse load may be determined using the Kinematic Theorem (i.e., an upper bound approach) by investigating the different possible collapse mechanisms available to the slab and seeking the mechanism which requires the least load to cause collapse. If all possible mechanisms have been examined then this minimum load will be the collapse load of the slab. The collapse load given by an assumed mechanism may be found by the Principle of Virtual Work or from the equations of equilibrium. Consider the two methods:

(i) Principle of Virtual Work: Let a slab which has been reduced to a collapse mechanism be given a small known virtual displacement in the direction of the collapse load. Then the virtual displacements of all points of the slab and the virtual rotations of the slab segments about the yield lines may be found in terms of the known virtual displacement and the dimensions of the segments. Consider the work done due to the virtual displacements and rotations.

The virtual work done by the internal actions of the slab will be due only to the yield moments at yield lines, since the virtual work done by the twisting moments and the shear forces at yield lines is zero when summed over the whole slab. The virtual work done by yield moment per unit width \( M_x \) which is acting in direction at angle \( \alpha \) to the \( x \) axis along the yield line of length \( L \) (at angle \( \alpha \) to \( y \) axis) at which the virtual rotation is \( \theta_x \) is:

\[ -\int_0^L M_x \theta_x \, dx \]

The work done is negative since the yield moment will be acting in the opposite direction to the virtual rotation. Substituting for \( M_x \) from equation (1), the virtual work done along the yield line is:

\[ -\int_0^L (M_x \cos^2\alpha + M_y \sin^2\alpha) \theta_x \, dx \]

(3)

In which the integrations are made along each yield line and the contributions from all yield lines are summed.

The virtual work done by the load per unit area at collapse, \( w_x \), when the slab undergoes virtual displacement \( \delta(x, y) \) is:

\[ w_x = \int \delta(x, y) \, dx \]
\[ \int \int w_u (x, y) \, dx \, dy - \int \left( \int M_x \, 0 \, dy + \int M_y \, 0 \, dx \right) = 0 \quad \ldots \quad (4) \]

In which the integrations are carried out over the whole area of the slab. It is to be noted that expression (4) represents the total external load multiplied by the movement of its centroid. Hence writing the equation of virtual work:

\[ \int \int w_u (x, y) \, dx \, dy - \sum (\int M_x \, 0 \, dy + \int M_y \, 0 \, dx) = 0 \quad \ldots \quad (5) \]

The statement of equation (5) is simply that when the slab in the collapse condition is given a small displacement, the external work done by the load on the slab in moving through that displacement is equal to the internal work done by the yield moments in moving through the rotation of yield lines due to the displacement. When equation (5) is applied to a particular slab the virtual displacement term, \( \delta \), cancels from the equation and the total load at collapse \( W_u \) is given in terms of the yield moments, slab dimensions and variables \( L_1, L_2, \ldots \) etc., which define the positions of the yield lines. The value of \( L_1, L_2, \ldots \) etc., which give the least load to cause collapse and hence the collapse load, may be found by solving simultaneously the following equations:

\[ \frac{\delta W_u}{\delta L_1} = \frac{\delta W_u}{\delta L_2} = \ldots = 0 \quad \ldots \quad (6) \]

(ii) Equations of Equilibrium: Each slab segment must be in equilibrium under the action of external loading, the actions at yield lines and, if the segment is at a supported edge of the slab, the reactions at supports. Usually the nodal forces within the slab are zero since all the internal yield lines are sagging. Further, the \( x \) and \( y \) direction components of the bending and twisting moments at yield lines are the \( x \) and \( y \) direction yield moments respectively, since there are no twisting moments in the \( x \) and \( y \) directions. In most cases, therefore, if the equilibrium equation for a segment is written by equating to zero the sum of the moments of all actions about the support line, the only actions that need to be considered are the yield moments and external loading. This equation will also include the dimensions which define the positions of the yield lines.

![Fig. 8](image)

If the equilibrium equations written for all segments are solved simultaneously the positions of the yield lines may be defined and the collapse load and yield moments related.

3.2. Analysis of Uniformly Loaded Two-Way Rectangular Slab with Fixed Edges

Consider the case of a slab with reinforcement placed in the \( x \) and \( y \) directions. Let the sagging yield moments of resistance at the edges be \( M_x \) and \( M_y \) and the hogging yield moments of resistance at the edges be \( M'_x \), \( M'_x \), \( M'_y \) and \( M'_y \) as shown in Fig. 8. The yield line pattern shown in Fig. 8 will form at collapse. The geometry of deformation shows the yield line EF to be parallel to the \( y \) axis. The positions of the yield lines are defined by \( L_1, L_2 \) and \( L_3 \).

(i) By Virtual Work: Let the slab be given a downward virtual displacement \( \delta \) at the yield line EF. Then the virtual rotations of segments in the \( x \) and \( y \) directions are:

- For segment ABFE: \( 0_x = \frac{\delta}{L_1} \quad 0_y = 0 \)
- For segment DCFE: \( 0_x = \frac{\delta}{L_2-L_1} \quad 0_y = 0 \)
- For segment ADE: \( 0_x = 0 \quad 0_y = \frac{\delta}{L_3} \)
- For segment BCF: \( 0_x = 0 \quad 0_y = \frac{\delta}{L_3} \)

Therefore virtual work done by yield moments is from expression (3):

\[ w_u \left[ \frac{\delta}{L_y (L_2 + L_3)} + \frac{\delta}{L_x (L_y - L_2 - L_3)} \right] = \frac{w_u \delta L_x}{6} (3L_y - L_2 - L_3) \quad \ldots \quad (8) \]

Now from equation (5), expressions 7 + 8 = 0, and hence

\[ w_u = \frac{L_y}{L_2} \left[ \frac{L_y}{L_1} + \frac{L_y}{L_1-L_2} \right] + \frac{L_x}{L_2} \left[ \frac{L_x}{L_1} + \frac{L_x}{L_1-L_2} \right] \quad \ldots \quad (9) \]
The values of \( L_1, L_2 \) and \( L_3 \) for minimum \( w_u \) can be found by solving simultaneously the equations:

\[
\frac{\partial w_u}{\partial L_1} = \frac{\partial w_u}{\partial L_2} = \frac{\partial w_u}{\partial L_3} = 0
\]

The collapse load is then calculated by substituting the determined values for \( L_1, L_2 \) and \( L_3 \) into equation (9).

The algebra involved in this method in this case is lengthy.

(ii) By Equations of Equilibrium: Segment ABFE, for sum of moments about axis AB to be zero:

\[
(M_x + M'_x) L_y^* = w_u \left[ \frac{L_1}{2} + \frac{(L_y - L_2 - L_3)L_1}{2} \right] = \frac{w_u L_1^2}{6} (3L_y^* - 2L_2 - 2L_3)
\]

Segment DCFE, for sum of moments about axis DC to be zero:

\[
(M_x + M'_x) L_y = \frac{w_u [(L_2 + L_3)(L_x - L_1)^2 + (L_y^* - L_2 - L_3)(L_x - L_1)^2]}{6} = \frac{w_u (L_x - L_1)^2}{6} (3L_y^* - 2L_2 - 2L_3)
\]

Segment ADE, for sum of moments about axis AD to be zero:

\[
(M_y + M'_y) L_x = \frac{w_u L_x L_2^2}{6}
\]

Segment BCF, for sum of moments about axis BC to be zero:

\[
(M_y + M'_y) L_x = \frac{w_u L_x L_2^2}{6}
\]

L_1, L_2, L_3 and \( w_u \) for a given slab may be found by solving equations (10), (11), (12) and (13) simultaneously.

It can be shown that \( L_3 \) is given by the solution of the quadratic

\[
\left( \frac{M_x + M'_x}{M_y + M'_y} \right) \left( 1 + \sqrt{\frac{M_x + M'_x}{M_y + M'_y}} \right)^2 \left( \frac{L_3}{L_x} \right)^2 + 2 \left( 1 + \sqrt{\frac{M_x + M'_x}{M_y + M'_y}} \right) \frac{L_3}{L_y} \frac{L_1}{L_x} - 3 = 0
\]

And

\[
L_2 = \sqrt{\frac{M_x + M'_y}{M_y + M'_y}} L_3
\]

The collapse load may then be found from either equation (12) or (13).

In extreme cases it may be found that \( L_2 + L_3 > L_y \). This means that the yield line pattern of Fig. 8 is incorrect and the correct pattern in this case has the sagging yield line EF running parallel to the x axis, i.e., the yield line pattern is turned through 90°.

If the slab has a simply-supported edge then a hogging moment yield line will not form along that edge and the hogging yield moment is put equal to zero in the expressions derived for fixed edges, e.g., if edges AB and BC were simply supported then \( M'_x = M'_y = 0 \).

4. DESIGN OF UNIFORMLY LOADED TWO-WAY RECTANGULAR SLAB BY YIELD-LINE THEORY

4.1. Determination of Yield Moments:

The problem in design is generally to determine the yield moments required for a slab with known dimensions, support conditions and collapse load, the collapse load being the working load multiplied by the required load factor. In design the yield-line pattern which requires the maximum yield moments to sustain the collapse load is sought. This follows since if reinforcement for smaller yield moments is provided the slab will fail at a load which is lower than the required collapse load. An unfortunate aspect of the upper bound approach used in yield-line theory is that failure to design to the correct yield-line pattern results in a slab with a lower load factor than expected. Fortunately, however, by using the expression for collapse load given by the virtual work equation it may be shown that a yield-line pattern which varies slightly from the true yield-line pattern leads to only a small variation in the collapse load. In the general case of a uniformly loaded two-way slab with various boundary conditions equation (9), with the exact values for \( L_1, L_2 \) and \( L_3 \) substituted, is lengthy. If, however, it is assumed that the
sagging moment yield lines running into the corners of the slab are at 45° to the x and y axes (i.e., \( L_1 = L_2 = \frac{1}{2} L_x \) in Fig. 8, if \( L_y > L_x \)) equation (9) is much simplified and yield moments are obtained which, in the case of well designed slabs, give a collapse load which is within a few percent of the true collapse load. On the basis of the 45° corner lines the relationship between yield moments and collapse load of a uniformly loaded rectangular two-way slab with reinforcement in the x and y directions such that \( M'_x = M'_x \) and \( M'_y = M'_y \), and with edges either fixed or simply supported, is, from equation (9):

If \( L_y > L_x \),

\[
\frac{w_u}{12} L_x^2 \left( 3 \frac{L_y}{L_x} - 1 \right) = 2 \left( \frac{L_y}{L_x} M_x + M_y \right) + K \quad . \quad (16)
\]

where K depends upon the support conditions and is given below for the various cases shown in Fig. 9.

Table I compares the accuracy of this assumption for practical slabs which have \( M'_x = M'_x \) and \( M'_y = M'_y \), with \( M'_{x} = \frac{2}{L_x} \) and various \( \frac{L_y}{L_x} \) and \( \frac{M_x}{M_y} \) ratios and support conditions, and shows that the accuracy of the assumption is reasonable for most cases. The accuracy obtained when using equation (16) could be improved if the design collapse load were to be increased by the percentage error indicated in Table I for the appropriate case.

The cases of fixed and simply supported slab edges are applicable to the design of slab and beam floors. The condition of a fixed edge arises when a panel is continuous with an adjacent panel at a supporting beam since continuity at a support is sufficient to justify treatment as a fixed edge in plastic theory. The outside edges of exterior panels of slab and beam floors are generally considered to be simply supported when obtaining moments in the remainder of the panel, and it would seem preferable to use this approach in plastic theory also. Having designed such a panel, however, some hogging moment steel would have to be placed along the outside edge to provide for the moment that will inevitably develop there due to the restraint of the walls and torsional stiffness of the supporting beam.

### Table 1.

**Accuracy of Assumption of 45° Corner Yield Lines for Uniformly Loaded Two-Way Rectangular Slabs**

<table>
<thead>
<tr>
<th>( \frac{L_y}{L_x} )</th>
<th>( \frac{M_x}{M_y} )</th>
<th>( \frac{L_y}{L_x} )</th>
<th>( \frac{M_x}{M_y} )</th>
<th>( \frac{M'_x}{M'_y} )</th>
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4.2. Cut-off Points of Hoggimg Moment Reinforcement:

Although the steel placed in the bottom of a slab for sagging moment must be placed without curtailment over the whole of the slab when designing by yield-line theory, (otherwise an alternative collapse mode will develop leading to a lower load factor), the steel placed in the top of the slab for hogging moment may be curtailed. To illustrate how the amount of curtailment may be determined the minimum length of the top steel for a uniformly loaded slab with all edges fixed and with 

\[ M' = \frac{M}{X} \]

and 

\[ M'' = \frac{M}{Y} \]

will now be calculated. Fig. 10 shows the reinforcement placed in such a slab. The lengths of top steel as a proportion of the spans is defined by the coefficient \( \lambda \), and \( \lambda \) will be taken to be the same for both spans.

Fig. 11 (a) shows the normal type of yield line pattern for such a slab. Fig. 11 (b) shows the alternative type of yield line pattern if the top steel is not continued over the whole of the slab. In the alternative type of failure the tensile strength of the concrete is neglected and the interior portion of the slab collapses as if simply supported along the lines of cut-off of the top steel.

From equation (16) the collapse load of the mechanism shown in Fig. 11 (a) is:

\[ W_u = \frac{24}{L_x^2(1-2\lambda)^2} \left( \frac{1}{\mu} \right) \left( \frac{L_y}{M_x} \right) \left( \frac{M_x + M_x'}{M_x + M_x'} \right) \]

(17)

And the collapse load of the mechanism shown in Fig. 11 (b) is:

\[ W_u = \frac{24}{L_x^2(1-2\lambda)^2(3-1)} \left( \frac{L_y}{M_x} \right) \left( \frac{M_x + M_x'}{M_x + M_x'} \right) \]

(18)

To examine the effect of varying the yield moments let

\[ \mu = \frac{M_x}{M_x'} \quad \text{and} \quad \gamma = \frac{M_y}{M_y'} \]

On substituting for \( M_x', M_y' \) and \( M_y \) in terms of \( M_x, \gamma \) and \( \mu \), equation (17) becomes:

\[ W_u = \frac{24}{L_x^2(1-2\lambda)^2} \left( \frac{1}{\mu} \right) \left( \frac{L_y}{M_x} \right) \left( 1 + \gamma \right) M_x \]

(19)

and equation (18) becomes:

\[ W_u = \frac{24}{L_x^2(1-2\lambda)^2(3-1)} \left( \frac{L_y}{M_x} \right) \left( 1 + \gamma \right) M_x \]

(20)

For failure of the type shown in Fig. 11 (a) to occur \( W_u \) from equation (19) must be less than or equal to that from equation (20) and hence:

\[ (1+\gamma) < \frac{1}{(1-2\lambda)^2} \]

In the limit \( (1+\gamma)(1-2\lambda)^2 \) = 1

and hence \( \lambda^2 - \frac{\gamma}{4(1+\gamma)} = 0 \)

The solution of this quadratic gives

\[ \lambda = 0.5 \left( 1 - \frac{\gamma}{\sqrt{1+\gamma}} \right) \]

(21)

The \( \lambda \) values for top reinforcement at the fixed edges of slabs with one, two or three simply supported edges can be calculated by a similar analysis. In these cases the interior portion which may collapse as a simply-supported slab will extend to the simply-supported edge or edges and \( \lambda \) is found to vary with both the \( L_x \) and \( M_x \) ratios in the general case. Table 2 gives the values for \( \lambda \) calculated for practical slabs in which \( M_x' = M_x', M_y' = M_y' \) and \( \frac{M_x}{M_x'} = \frac{M_y}{M_y'} = 2.0 \) for the cases of support conditions shown in Fig. 9. The length of hogging moment reinforcement (measured from support to theoretical cut-off points) in the \( x \) and \( y \) directions at fixed edges is \( \lambda L_x \) and \( \lambda L_y \) respectively. At a simply supported edge \( \lambda \) is zero. It is to be noted that bond length must be provided in addition to the above lengths to ensure that the bars are anchored satisfactorily.

Some authorities, e.g. Wood (6), maintain that there is another mode of collapse which may govern the length of hogging moment reinforcement in panels of a continuous slab and beam floor. This mode may occur when live load is placed only on alternate panels to give a "chequered" live load distribution. Then the panels which carry dead load only may be forced up and failed if the top steel is not carried far enough into the panel. This mode of collapse is of no concern in practical
and increases from unity the value 

\[ \frac{2 \lambda L_x V}{L_y} = - \lambda \]

On solving equations (21), the volume of steel becomes:

\[ V = \frac{w_u L_x L_y}{24(1+\gamma)} \left[ 2 \frac{L_x}{L_y} - 2L_x \sqrt{\left( \frac{L_x}{\mu L_y} \right)^2 + \frac{3}{\mu}} \right] \]

On substituting \( M_x \) from equation (23) into equation (22)

\[ V = \frac{w_u L_x L_y}{24(1+\gamma)} \left[ 1 - \gamma(1+\gamma)^{3/2} \right] \left( 1+\frac{L_x}{L_y} \right) - \frac{2L_x}{L_y} \sqrt{\left( \frac{L_x}{\mu L_y} \right)^2 + \frac{3}{\mu}} \]

For a slab with given dimensions and collapse load, equation (24) gives the volume of steel in terms of \( \gamma \) and \( \mu \). The values of \( \gamma \) and \( \mu \) for minimum volume of steel will now be examined:

(i) Considering \( \frac{\partial V}{\partial \gamma} = 0 \) from equation (24) gives \( \gamma = 2 \) for minimum \( V \).

(ii) Considering \( \frac{\partial V}{\partial \mu} = 0 \) from equation 24 gives the following values of \( \mu \) for minimum \( V \) for slabs with various \( \frac{L_y}{L_x} \) ratios:

<table>
<thead>
<tr>
<th>( \frac{L_y}{L_x} )</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
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<td>3.9</td>
<td>5.7</td>
<td>7.7</td>
<td>10.0</td>
<td>16.7</td>
<td>25.0</td>
</tr>
</tbody>
</table>

Fig. 12 shows the variation of \( V \) with \( \mu \) and it can be seen that as \( \frac{L_y}{L_x} \) increases from unity the value of \( \mu \) for minimum \( V \) increases rapidly and soon results in a value which cannot be used in practice due to the Code requirement for minimum steel in one direction. (According to B.S.C.P.P. 114 the minimum reinforcement in either direction should be not less than 0.15% of the design, however, since in order for it to occur, the supporting beams have to fail in torsion (this point has been neglected by Wood), and also the ratio of live load to dead load must be higher than normal.

4.3. Distribution of Reinforcement for Minimum Weight of Steel

The equations relating yield moments and collapse load give a measure of the sum of the yield moments required for a particular slab. The allocation of reinforcement for the various yield moments is best made by considering the conditions for minimum amount of steel. Consider the case of a uniformly loaded slab with all edges fixed and with \( M''_x = M'_x \) and \( M''_y = M'_y \). Let the reinforcement be placed as in Fig. 10. The areas of steel per unit width are given by \( A'_{ix} = \beta M'_x \), \( A'_{iy} = \beta M'_y \), \( A''_{ix} = \beta M''_x \), \( A''_{iy} = \beta M''_y \), where \( \beta \) is the reciprocal of the product of the yield force of the steel and the lever arm. Strictly, \( \beta \) will not have the same value for each yield moment, but the assumption of constant \( \beta \) is sufficiently accurate for a study of economy. Examination of Fig. 10 shows that the volume of steel in the slab (neglecting the bond length of the hogging moment steel) is:

\[ V = \beta L_x L_y [M_x + M_y + 2(x(M''_x + M''_y))] \]

To examine the effect of varying the ratio of yield moments let

\[ \mu = \frac{M_x}{M_y} = \frac{M''_x}{M''_y} \]

and \( \gamma = \frac{M'_x}{M'_y} \). On substituting for \( M''_x, M''_y \) and \( M_y \) in terms of \( M_x, \mu \) and \( \gamma \), and putting \( \lambda \) equal to the value given by equation (21), the volume of steel becomes:

\[ V = \beta L_x L_y (1 + \frac{1}{\mu}) [1 - \gamma + \gamma(1+\gamma)^{-1}]M_x \]

Also for the slab, on solving equations (10), (11), (12) and (13) simultaneously, and putting all yield moments in terms of \( M_x, \mu \) and \( \gamma \), the value of \( M_x \) is found to be:

\[ M_x = \frac{w_u L_x L_y}{24(1+\gamma)} \left[ 2 \frac{L_x}{L_y} - 2L_x \sqrt{\left( \frac{L_x}{\mu L_y} \right)^2 + \frac{3}{\mu}} \right] \]

TABLE 2.

Coefficients Defining Lengths of Hogging Moment Reinforcement at Fixed Edges of Uniformly Loaded Two-Way Rectangular Slabs

<table>
<thead>
<tr>
<th>( \frac{L_y}{L_x} )</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
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<td>( \mu )</td>
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gross cross-sectional area of concrete.) Hence most rectangular slabs cannot be designed for maximum economy of steel and for such slabs the best available design is obtained when \( \mu \) is as large as possible, i.e., when minimum steel is placed in the direction of the long span.

A further aspect of design by plastic theory is that the distribution of reinforcement should not differ too greatly from that given by elastic theory otherwise cracking at working load may be excessive. It is fortunate that a value of \( \gamma = 2 \) satisfies both economy and elastic theory (approximately) for slabs with all edges fixed.

Although in the above only the case of a slab with all edges fixed has been considered, it is evident that for slabs with various combinations of fixed and simply supported edges in order to control cracking \( \gamma \) should not vary much from 2 and for rectangular slabs for maximum economy in most practical cases \( \mu \) should be made as large as possible.

4.4. Example of Design According to B.S.C.P.114(1957) by Yield-Line Theory

(i) Data: An interior panel of a slab and beam floor spans between beams on the lines of a 16 ft \( \times \) 24 ft column grid and is to be designed using concrete with a 28 day works cube strength of 3,000 lb/in\(^2\) and mild steel to B.S. 785 with no guaranteed yield stress. Live load and partitions total 80 lb/ft\(^2\).

(ii) Design: Using a 5 in-thick slab (> 1/6 of span) dead load of slab and finishes totals 80 lb/ft\(^2\). Total working load is 160 lb/ft\(^2\). The Code requires a minimum load factor of 2 to be used and hence the design collapse load, \( w_u \), is 320 lb/ft\(^2\). In order to calculate the yield moments, for under-reinforced section the Code gives:

\[
\text{Yield moment per unit width} = 2A_{st}P_{st}(d) \left( \frac{3A_{st}P_{st}}{4\rho_{cb}} \right). \tag{25}
\]

The above equation from the Code was derived from ultimate load considerations and not elastic theory, although the stresses in steel and concrete at ultimate load are expressed in terms of working stresses. The yield moment per unit width from equation (25) may not exceed 0.5 \( \rho_{cb}d^2 \), without compression steel being placed. In this example \( P_{st} = 20,000 \text{ lb/in}^2 \), and \( \rho_{cb} = 1,000 \text{ lb/in}^2 \).

Say \( 1/3 \) in-diameter bars with \( 1/2 \) in cover are used in \( x \) direction and 5/16 in-diameter bars are used in \( y \) direction. Effective depths to steel are 4.31 in and 3.97 in in \( x \) and \( y \) directions respectively. In accordance with the findings of section 4.3, let \( M_y \) be due to minimum amount of steel (0.15% of gross area of concrete), \( M'_x = 2M_x \) and \( M'_y = 2M_y \).

Then \( A_{st y} = 0.0015 \times 12 \times 5 = 0.090 \text{ in}^3/\text{ft width} \). Hence from equation (25), \( M_y = 13,900 \text{ lb in ft width} \).

\[ M_y = 2M_y = 27,800 \text{ lb in ft width} \]

To determine \( A'_{st y} \) substitute the value of \( M'_y \) into equation (25) and solve the quadratic, then \( A'_{st y} = 0.186 \text{ in}^3/\text{ft width} \).

Now from equation (16):

\[
\frac{w_u}{L_x^2}(\frac{L_y}{L_x} - 1) = \frac{2}{12} \left[ \frac{L_y}{L_x} (M_x + M'_x) + (M_y + M'_y) \right]
\]

On substituting known quantities into the above equation and putting \( M'_x = 2M_x \), the remaining yield moments are found:

\[ M_x = 22,600 \text{ in lb/ft width} \]

Substituting \( M_x \) and \( M'_x \) into equation (25) in turn, the remaining steel areas are found:

\[ A_{st x} = 0.137 \text{ in}^3/\text{ft width} \]

Top steel may be curtailed. For this design \( \gamma = 2 \) and from equation (21) \( \lambda = 0.212 \). If a bond length of 12 in-diameter bars is used in addition, then lengths of top steel at each side of the slab in \( x \) and \( y \) directions respectively are 3.77 ft and 5.40 ft.

Total volume of steel used in the panel is 2,060 in\(^3\).

In section 4.4 it was shown that for \( \frac{1}{L_x} = 1.5 \) for maximum economy of steel \( \mu = 4.7 \). Obviously this value cannot be used in the above example and the maximum possible value \( (\mu = 1.63) \) was used.

(iii) Comparison with Conventional Design: If the panel is designed using the coefficients of Table 17 of B.S.C.P. 114 it may be shown that the areas of steel required at sections of maximum bending moment in the middle strips are:

\[ A_{st x} = 0.279 \text{ in}^3/\text{ft width} \]

\[ A_{st y} = 0.378 \text{ in}^3/\text{ft width} \]

\[ A'_{st x} = 0.163 \text{ in}^3/\text{ft width} \]

\[ A'_{st y} = 0.220 \text{ in}^3/\text{ft width} \]

If in the middle strips, half the bottom steel is bent up at points of contraflexure and the top steel is curtailed at bond distance beyond the points of contraflexure; and if in the edge strips the minimum amount of steel allowable is placed, it may be shown that the total volume of steel required is approximately 9% greater than that for the yield-line theory design. Also the conventional design requires much closer spacing of reinforcement in middle strips than does the yield-line theory design. If the loading on the slab had been greater the yield-line theory design would have been more economical as a greater value of \( \mu \) could have been used.
YIELD-LINE THEORY (Continued from p. 64.)

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