Dynamically dual vibration absorbers: a bond graph approach to vibration control

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Dynamically dual vibration absorbers: a bond graph approach to vibration control

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This paper investigates the use of an actuator and sensor pair coupled via a control system to damp out oscillations in resonant mechanical systems. Specifically the designs emulate passive control strategies, resulting in controller dynamics that resemble a physical system. Here, the use of the novel dynamically dual approach is proposed to design the vibration absorbers to be implemented as the controller dynamics; this gives rise to the dynamically dual vibration absorber (DDVA). It is shown that the method is a natural generalisation of the classical single-degree-of freedom mass–spring–damper vibration absorber and also of the popular acceleration feedback controller. This generalisation is applicable to the vibration control of arbitrarily complex resonant dynamical systems. It is further shown that the DDVA approach is analogous to the hybrid numerical-experimental testing technique known as substructuring. This analogy enables methods and results, such as robustness to sensor/actuator dynamics, to be applied to dynamically dual vibration absorbers. Illustrative experiments using both a hinged rigid beam and a flexible cantilever beam are presented.

Keywords: vibration absorber; bond graph; acceleration feedback control; dynamic dual

1. Introduction

The use of a secondary resonant mechanical systems to damp out oscillations in a resonant mechanical system by absorbing and dissipating energy has a long history and early work is summarised in the classical textbook by Den Hartog (1985). An alternative method for damping unwanted oscillations is to use some form of active vibration control. To achieve this some type of actuator and sensor system needs to be used. For example, vibrations can be damped from a mechanical system using a piezo-electric transducer and an associated electrical circuit (Hagood & von Flotow, 1991). This can have considerable advantages, although, as discussed by Moheimani & Behrens (2004) multi-modal resonant structures require sophisticated circuit synthesis.

The adjective ‘passive’ applied to ‘system’ has two different but related meanings: a physical system not containing a power source and a mathematical expression imposing the corresponding property on the input and output variables of a set of equations (Hogan, 1985; Sharon, Hogan, & Hardt, 1991; Slotine & Li, 1991). In general, this means that passive mechanical (or electrical) vibration absorbers can be replaced by a computer and associated sensor-actuator pairs which emulate the physical passivity in the equivalent mathematical sense. The algorithm implemented in the computer does not have to represent a physical system and can be designed using conventional control-theoretic methods (Balas, 1978; Fleming & Moheimani, 2005; Hong & Bernstein, 1998; Hogsberg & Krenk, 2006; Moheimani & Fleming, 2006), optimisation (Krenk & Hogsberg, 2009) or via system inversion (Ali & Padhi, 2009).

However, it can be argued that there are advantages in implementing physical systems within the digital computer (Gawthrop, 1995; Gawthrop, Bhikkaji, & Moheimani, 2010; Hogan, 1985; Lozano, Brogliato, Egeland, & Maschke, 2000; Ortega, Loria, Nicklasson, & Sira-Ramirez, 1998; Ortega, van der Schaft, Mareels, & Maschke, 2001; Sharon et al., 1991; Slotine & Li, 1991); this is the approach explored in this paper. In particular, the well-known relationship between dissipativity, passivity and physical systems (Lozano et al., 2000; Ortega et al., 1998, 2001; Willems, 1972) is exploited. Such energy based concepts rely on the properties of physical connections. In particular, the concept of collocation is a key system property in the context of active vibration control (Gawthrop et al., 2010; Preumont, 2002).

Replacing a mechanical vibration absorber by a digital computer is analogous to the well-known hybrid numerical-experimental testing technique where the structure under consideration is split into an experimental test piece (or physical substructure) and a numerical model.
describing the remainder of the structure (or numerical substructure). Although the two coupled passive sub-
systems resulting from this process are stable (Anderson & Vongpanitlerd, 2006; Desoer & Vidyasagar, 1975;
Lozano et al., 2000; Ortega, Praly, & Landau, 1985; Ortega et al., 2001), this stability can be destroyed by the
digital implementation of the numerical substructure and the corresponding actuator and sensor dynamics that
couple the substructures (Gawthrop, Wallace, Neild, & Wagg, 2007). Fortunately, this problem of coupling the
numerical and experimental substructures using real-time digital implementation has been solved and a suite of
techniques for robust numerical-experimental substructuring is now available (Blakeborough, Williams, Darby,
& Williams, 2001; Gawthrop, Wallace, & Wagg, 2005; Wagg, Neild, & Gawthrop, 2008).

As discussed by Gawthrop et al. (2005), the bond-graph approach (Borutzky, 2011; Gawthrop & Bevan, 2007;
Gawthrop & Smith, 1996; Karnopp, Margolis, & Rosenberg, 2012; Mukherjee, Karmaker, & Samantaray, 2006)
gives a natural and convenient formulation of substructuring and control (Gawthrop, 2004; Gawthrop et al., 2005;
Vink, Ballance, & Gawthrop, 2006) and the concept of actuator/sensor collocation has a clear bond graph inter-
pretation. For these reasons, the bond-graph approach is adopted in this paper.

As discussed by Den Hartog (1985), choosing the structure of a vibration absorber for a single degree of freedom
system is straightforward. However, choosing the structure for multi-degree of freedom systems such as those arising
from modal decomposition is considerably more involved (Moheimani & Behrens, 2004). This complexity motivates
the novel approach of this paper based on the concept of of a dynamically dual \(1\) system.

As discussed in more detail in Section 3, a dynamically dual mechanical system is obtained by interchanging the rôles of velocity and force. This concept of duality has been used for analysis of dynamical systems (Cellier, 1991; Karnopp, 1966; Samanta & Mukherjee, 1985, 1990; Shearer, Murphy, & Richardson, 1971), and this paper uses the concept to design dynamically dual vibration absorbers (DDVA).

Although the DDVA method originated an extension of the physically based design of Den Hartog (1985), it will be shown that the method also includes the well-established acceleration feedback approach (Preumont, 2002).

In summary, placing both the traditional Den Hartog mechanical vibration absorber and acceleration feedback into the wider context of the DDVA of this paper has two advantages: the method immediately extends to multi-

degree of freedom systems and the implementation and theoretical results (including robustness to sensor/actuator
dynamics) from substructuring can be directly applied.

Section 2 reviews the substructuring approach to provide a framework for the paper. Section 3 gives the
foundations of the DDVA approach; Section 3.2 focuses on the Den Hartog (1985) absorber version and Section 3.3
focuses on the acceleration feedback controller (Preumont, 2002). Section 4 discusses a number of multi-mode
examples. Section 5 gives illustrative experimental results; Sections 5.1 and 5.2 experimentally verify the approach when
applied to a rigid beam with a flexible joint and a flexible cantilever beam, respectively. Section 6 concludes the
paper.

2. The substructuring formulation

Substructuring is a novel dynamic testing technique that allows the experimental testing of a component within
the context of a larger system. This is achieved through the coupling of the physical component with a controller
that numerically simulates the dynamics of the remainder of the system. Note that as the controller dynamics are
designed to simulate part of a real system, the dynamics are physically realisable.

Figure 1 summarises the basic substructuring formulation Gawthrop et al. (2005) and Gawthrop, Wagg, & Neild
(2009). For simplicity, Figure 1 will be assumed to represent a system with scalar quantities, although this can readily be extended to vectors. The three key parts are shown in Figure 1:

1. **\( Phy \)** representing the physical component, with transfer function \( p(s) \), to be controlled,
2. **\( Num \)** representing the controller, with physically realisable dynamics, which is implemented numerically and has a transfer function \( n(s) \),
3. **\( Se:F_0 \)** representing a disturbing external force, \( F_0 \),

where \( s \) is the Laplace domain independent variable. Firstly the velocity feedback case is shown, Figure 1(a) as a bond graph\(^2\) and in Figure 1(b) as a block diagram. Here, the physical component \( Phy \) has a force input, \( F_0 - F \), and a measured velocity output, \( v \). In addition, the parameters in the physical system are represented by a vector, \( \theta_p \). Similarly, \( \theta_n \), represents the vector of parameters in the numerical system.

An advantage of the bond graph representation is that it emphasises the fact that the physical system \( Phy \) and the controller \( Num \) are connected by power bonds and thus the control system is collocated — meaning that the actuator and sensor are located at the same point. In Figure 1(a) the parts are joined by a common flow (velocity) junction denoted as 1. The bond graph also indicates causality and \( Phy \) and \( Num \) are represented by the positive real transfer functions \( p(s, \theta_p) \) and \( n(s, \theta_n) \), respectively. The transfer
functions are related by the following relationships:

\[ v = p(s, \theta_p)(F_0 - F), \quad (1) \]
\[ F = n(s, \theta_n)v. \quad (2) \]

Although it is natural to work in terms of velocity \( v \) rather than displacement \( x \), Figure 1(b) can be easily rewritten in terms of displacement as Figure 1(c) where \( a = dv/dt \) or in terms of acceleration as Figure 1(d) where \( \ddot{a} = d^2v/dt^2 \). The choice of formulation (displacement, velocity or acceleration) does not change the theoretical closed-loop stability properties defined by the loop-gain \( L(s) = n(s, \theta_n)p(s, \theta_p) \), but allows flexibility in the choice of sensor and actuator. As well as providing a conceptual basis for this paper, the substructuring approach links to classical control system concepts useful for stability and robustness analysis. Details can be found elsewhere (Gawthrop et al., 2007, 2009; Wagg et al., 2008).

The substructuring formulation of Figure 1(a) assumes an inertia-like physical component driven by a force; as discussed by Gawthrop et al. (2005), compliance-like physical components can be treated by the formulation of Figure 2(a) where the external force \( F_0 \) is replaced by an external velocity \( v_0 \) and the three components are now connected by a common force, or \( 0 \), junction. To distinguish this velocity-driven formulation from the force-driven formulation in Figure 1(a) an over-bar is used:

\[ F = \bar{p}(s, \theta_p)(v_0 - v), \quad (3) \]
\[ v = \bar{n}(s, \theta_n)F. \quad (4) \]

Using the definitions of Equations (3) and (4), the block diagram equivalent of Figure 2(a) is Figure 2(b). Once again, displacement and acceleration formulations are given by Figure 2(c) and 2(d), respectively.

3. Dynamically dual design

As already discussed, in this paper a vibration absorber attached to a system is considered. This vibration absorber, while based on a physical component thus ensuring that the system is passive, is implemented as a controller. This setup can be considered within the substructuring framework, with the vibration absorber forming \( \text{Num} \) and the system which the vibration absorber is attached being \( \text{Phy} \). One possible absorber is the Den Hartog resonant vibration absorber, which is usually represented by a conventional mass–spring–damper schematic. However, as pointed out by Den Hartog (1985), and discussed in greater depth by Shearer et al. (1971), can equally well be described by an equivalent electrical circuit analogue.

Here, the use of DDVAs is proposed as a method for generating suitable \( \text{Num} \) dynamics. As discussed in Section 3.2, the resonant vibration absorber of Den Hartog (1985) is an example of a DDVA and provides the motivation for this approach. In formulating the DDVA approach the following features of the Den Hartog resonant vibration absorber are abstracted and generalised:

(1) it is a one-port\(^3\) passive\(^4\) physical system,
(2) it is causally compatible with the system, in that, the output velocity of the system provides the input to the absorber and the force output of the absorber provides the input to the system,
(3) there is a variable coupling parameter,
(4) the absorber has the same resonant frequency as the system, and
(5) the damping ratio of the absorber is greater than that of system.

The DDVA design approach is to set \( \text{Num} \) to be a dynamic-dual of the key mode or modes of the system that...
the absorber is attached to (which is contained in Phy).
The method of obtaining a dynamic dual is now discussed.
This is followed by discussions of two common absorber strategies which are DDVA; the Den Hartog absorber and
the acceleration feedback method proposed by Preumont (2002).

3.1. A dynamic-dual
A dynamic-dual of a system is obtained by interchanging
the rôles of velocity and force, is defined in Shearer et al.
(1971). An extended version of this concept, the scaled
dual is used here and, in the context of mechanical systems
is defined as follows:

(1) Each force $F_i$, and each velocity $v_i$, in the original
system has a scaled dual $v^D_i$, and $F^D_i$ in the dual
system given by:

$$v^D_i = \frac{1}{g} F_i, \quad (5)$$
$$F^D_i = g v_i, \quad (6)$$

where $g$ is the scaling factor and $g = 1$ corresponds
to the unscaled dual.

(2) Each mass component with mass $m_i$ is replaced in
the dual system with a spring component of stiffness $K_i$, each spring component with stiffness $k_i$
is replaced in the dual system with a mass component
of mass $M_i$, and each damper component with damping
coefficient $r_i$ is replaced in the dual
system with a damper component with damping
coefficient $R_i$ where:

$$K_i = \frac{g^2}{m_i}, \quad (7)$$
$$R_i = \frac{g^2}{r_i}. \quad (9)$$

(3) Common force connections become common
velocity connections and common velocity
connections become common force connections in the
dual system.

(4) If the system transfer function $h(s)$ has force $F$ as
input and velocity $v$ as output, then the dual trans-
fer function $H(s)$ has velocity $v^D$ as input and force
$F^D$ as output and is given by:

$$H(s) = \frac{F^D}{v^D} = \frac{g v}{(1/g) F} = g^2 h(s). \quad (12)$$

Equations (10) and (11) are power conserving in the sense
that

$$F^D v^D = F v. \quad (13)$$

As these Equations (10) and (11) also interchange the roles
of force and velocity, they correspond to the bond graph
gyrator (GY) component of Figure 3.

Equations (7) and (8) ensure that the scaled dual retains
the same natural frequencies as the system; in the sequel,
the value $K_i$ given using Equation (9) is not used, instead it

Figure 3. Gyrator interpretation of dynamic-dual.
is replaced by the user-selected value \( R' \). This allows the damping of the modified scaled dual system, which forms the controller implemented in \( \text{Num} \), to be adjusted.

Although not essential to the approach of this paper, the bond graph formulation provides a clear exposition of the notion of a scaled dual. In particular, the scaled dual system can be described in two different but equivalent ways as:

1. The bond graph dual where the component moduli are given by Equations (7)–(9) or
2. Following Equation (13), the system obtained by appending a gyrator of modulus \( g \) to the system \( \text{Phy} \).

This point is also discussed by Gawthrop et al. (2010).

3.2. **Den Hartog absorber**

In his classical text book (Den Hartog, 1985, Section 3.3), Den Hartog considers the design of a damped vibration absorber for an undamped mass–spring oscillator which is subject to a force disturbance. The specifications were considered.

In the terminology used in this paper, the physical system requiring vibration suppression (\( \text{Phy} \)) is the undamped mass–spring oscillator. In Den Hartog (1985) the vibration absorber was considered to be a physical mechanical device but here it is considered to be a controller (with sensor and actuator) with the same dynamics as the absorber and forms \( \text{Num} \). The disturbance force acts on the undamped mass–spring oscillator \( \text{Phy} \), as does a force due to the presence of the absorber \( \text{Num} \), therefore the system can be represented by the block diagram given in Figure 1(b).

Figure 4(b) and 4(a) gives the schematic diagram of the damped vibration absorber \( \text{Num} \) and the undamped oscillator \( \text{Phy} \), respectively. A damper with \( r = \infty \) is included in the subsystem \( \text{Phy} \) of Figure 4(a) to allow for the corresponding component in the subsystem \( \text{Num} \).

Using standard manipulations, the transfer function of the physical system, \( \text{Phy} \), of Figure 4(a) is:

\[
p(s, \theta_p) = \frac{s(ms + r)}{mr^2 + kms + kr}.
\]  

(14)

Letting \( r \to \infty \) gives:

\[
p(s, \theta_p) = \frac{s}{ms^2 + k}.
\]  

(15)

Similarly, from Figure 4(b), the transfer function for \( \text{Num} \), which represents the Den Hartog absorber, is

\[
n(s, \theta_n) = \frac{Ms(Rs + K)}{Ms^2 + Rs + Kr}.
\]  

(16)

It can be shown that this absorber is a scaled dynamic-dual of the system, \( \text{Phy} \), it is applied to. Considering the system \( \text{Phy} \), given in Equation (14), and applying the dual transforms, given in Equations (7)–(9), the parameters \( m, k \) and \( r \) can be rewritten in terms of \( M, K \) and \( D \) to give:

\[
p(s, \theta_p) = \frac{s(g^2 + K)s + (g^2/R)}{(g^2/K)s^2 + (g^2/M)(g^2/K)s + (g^2/M)(g^2/R)} = \left( \frac{1}{g^2} \right) \frac{Ms(Rs + K)}{Ms^2 + Rs + Kr}.
\]  

(17)

Applying the scaling given in Equation (12), the scaled dual of \( \text{Phy} \) is

\[
P(s, \Theta_p) = g^2 p(s, \theta_p) = \frac{Ms(Rs + K)}{Ms^2 + Rs + Kr}.
\]  

(18)

Thus, by comparing this to Equation (16), it can be seen that the Den Hartog absorber in \( \text{Num} \) corresponds to the scaled dual of \( \text{Phy} \):

\[
n(s, \theta_n) = P(s, \Theta_p).
\]  

(19)

The first part of the Den Hartog specifications is achieved by setting:

\[
M = am \quad \text{where} \quad a = \frac{1}{20}.
\]  

(20)

The second part of the specification is achieved by setting

\[
K = \frac{k}{m}.
\]  

(21)

Equations (20) and (21) imply that

\[
K = \alpha k.
\]  

(22)

Moreover, using Equations (7) and (22), the scaling gain \( g \) is given by

\[
g^2 = Km = \alpha mk.
\]  

(23)
Finally, the third part of the specification is achieved by replacing the damping coefficient $R$ of $n(s, \theta_n)$ by

$$R' = 2\zeta \sqrt{MK}. \quad (24)$$

To illustrate the properties of this particular vibration absorber, the unit system with $m = k = 1$ was used. Using the specification described above, this gives the numerical system parameters $M = K = 0.05$ and $R' = 0.01$, and so the DDVA is given by:

$$n(s, \theta'_n) = \frac{Ms(R's + K)}{Ms^2 + R's + K} = \frac{0.01s^2 + 0.05s}{s^2 + 0.2s + 1}. \quad (25)$$

The corresponding closed-loop frequency response appears in Figure 4(c); this shows the ‘split peak’ phenomenon described by Den Hartog (1985). The corresponding closed-loop impulse response appears in Figure 4(d); this decays exponentially over the time scales determined by the specified damping ratio.

### 3.3. Acceleration feedback

The acceleration feedback method has been proposed by Preumont (2002). This section rederives the algorithm from the DDVA point of view. In particular, the undamped physical system of Figure 4(a) (with $1/r = 0$) can equally well be represented in Figure 5(a) with $r = 0$. This system has a different modified dual and thus gives a different form of control; this turns out to be a form of acceleration feedback.

As with the last example the vibration absorber is acting on an undamped mass–spring oscillator. The undamped oscillator forms $\text{Phy}$ as shown in Figure 5(a). Note that a damper with $r = 0$ is included to allow a dynamic-dual to be formulated. Using standard manipulations, the transfer function of the physical system $\text{Phy}$ of Figure 5(a) is

$$p(s, \theta_p) = \frac{s}{ms^2 + rs + k}. \quad (26)$$
Figure 5. Acceleration feedback. Like the Den Hartog absorber of Figure 4, the physical system and the dual are mass–spring–damper systems, but the configuration is different. The closed and open-loop responses are similar to those of the Den Hartog absorber of Figure 4, but the split peaks are more symmetrical. (a) Phy: the physical system, (b) Num: the dual physical system, (c) frequency response $H(j \omega)$ and (d) impulse response $h(t)$.

Setting $r = 0$ gives

$$p(s, \theta_p) = \frac{s}{ms^2 + k}. \quad (27)$$

The controller transfer function, forming Num, for the acceleration feedback method (Preumont, 2002) is given by

$$n(s, \theta_n) = g^2 p(s, \theta_p) = \frac{g^2 s}{ms^2 + rs + k}. \quad (28)$$

Using Equation (12), it can be seen that the numerical system Num is the scaled dynamic-dual of Phy. Figure 5(b) gives a physical representation of the acceleration feedback controller, where the component values $R, K$ and $M$ can be calculated using Equations (7)–(9) but are not required.

To give a direct comparison with Section 3.2, the same system and design considerations are used to give the DDVA:

$$n(s, \theta'_n) = \frac{0.05s}{s^2 + 0.2s + 1}. \quad (29)$$

This is similar to the DDVA of Equation (25) except that the numerator $s^2$ term is not present.

The corresponding closed-loop frequency response appears in Figure 5(c); this is similar to Figure 4(c) except that the peaks have a more similar amplitude. The corresponding closed-loop step response appears in Figure 5(d); again, this decays exponentially over the time scales determined by the specified damping ratio.

4. Systems with multiple modes of vibration

The examples discussed in the previous section demonstrate that the DDVA approach gives the same type of vibration absorber for the well known cases associated with mass–spring–damper systems. The real advantage of the DDVA approach is when using it to reduce vibrations in systems with multiple modes of vibration. The steps involved are the same as above: (i) define a physical model of the system Phy, (ii) set Num as the modified scaled dual of Phy, and (iii) connect the systems via a single (one-port) connection. As discussed in Section 3.1, step (ii) can either be accomplished directly or indirectly using the GY approach of Figure 3. In this paper, attention is focused...
on linear models thus giving rise to transfer-function representations.

Two examples are considered here. The first is a two degree-of-freedom lumped mass system which is shown schematically in Figure 6(a) and 6(b). Figure 6(a) and 6(b) are similar to Figure 5(a) and 5(b) except that there are two coupled mass–spring damper systems involved. Thus Phy can be regarded as the modal decomposition of a 2DOF system and Num the corresponding vibration absorber. For the purposes of illustration, each subsystem of Num has the same parameters as in the example of Figure 5 of Section 3.3 (Table 1).

Table 1. Modal system: resonant frequencies.

<table>
<thead>
<tr>
<th>n</th>
<th>wn (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Figure 6(c) shows the open (without the vibration absorber) and closed-loop (with the vibration absorber) frequency response magnitudes. The magnitude of the closed-loop response (black line) is clearly smaller than the corresponding open-loop response (grey line) at the two resonant frequencies. Figure 6(d) shows the equivalent impulse response; as predicted by the frequency responses, the closed-loop impulse response decays more rapidly than the equivalent open-loop response.

As a second example, a uniform Euler–Bernoulli cantilever beam (with one end fixed and the other free) modelled using a 10 element, finite-element bond graph model (Karnopp, Margolis, & Rosenberg, 2000; Margolis, 1985) is considered. Such beam models are undamped; but the DDVA approach needs to include damping in Num. For the purposes of this example, Rayleigh damping is assumed; in particular, each compliant element in the lumped model has an associated damping term represented by a damper connected across the ends of the compliant elements.

Following Balas (1978, Section V) who considers a ‘unit beam’, the cantilever beam is normalised to have unit mass per unit length and unit compliance per unit length. Phy is assumed to have a small (but non-zero) damping of $10^{-6}$ per unit length. The 10 modal frequencies appear in Table 2. For the purposes of illustration, the vibration absorber was applied to the beam using a collocated point Force/Velocity actuator/sensor halfway along the beam. As discussed in the sequel, this point corresponds to a nodal point of the third-resonance and thus this mode cannot be controlled with this choice. The choice of actuator/sensor location is an interesting topic not considered in this paper.

Table 2. Cantilever beam model modal frequencies.

<table>
<thead>
<tr>
<th>n</th>
<th>wn (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.919</td>
</tr>
<tr>
<td>2</td>
<td>18.28</td>
</tr>
<tr>
<td>3</td>
<td>50.37</td>
</tr>
<tr>
<td>4</td>
<td>95.59</td>
</tr>
<tr>
<td>5</td>
<td>150.4</td>
</tr>
<tr>
<td>6</td>
<td>210.0</td>
</tr>
<tr>
<td>7</td>
<td>269.1</td>
</tr>
<tr>
<td>8</td>
<td>322.0</td>
</tr>
<tr>
<td>9</td>
<td>363.9</td>
</tr>
<tr>
<td>10</td>
<td>390.8</td>
</tr>
</tbody>
</table>

Note: Not all frequencies appear in Figures 7 and 8 due to coincident zeros.

The second approach is to use the scaled dynamic-dual of a two mode modal model (as in Figure 6(b)), capturing the dynamics of the first two modes of the cantilever. This is then connected to the same cantilever beam. The parameters of Num are the same as in the example of Figure 6 and those of Phy the same as those of the example of Figure 7.

Figure 8(a) shows the open (without the vibration absorber) and closed-loop (with the vibration absorber) frequency response magnitudes. This figure has been expanded to show the frequency responses close to the first–fourth resonances in Figure 8(c)–8(f), respectively. Near the first two resonances (Figure 8(c) and 8(d)), the magnitude of the closed-loop response (black line) is clearly smaller than the corresponding open-loop response (grey line) at the two resonant frequencies. The third resonance corresponds to a node at the sensor/actuator and the fourth is well damped anyway. Thus this controller design naturally applies control authority at the important resonances. Figure 7(b) shows the equivalent impulse response; as predicted by the frequency responses, the closed-loop impulse response decays more rapidly than the equivalent open-loop response. As noted above, the third resonance is not controlled using this approach. However, it could be controlled either by moving the sensor/actuator away from the node or by having a second sensor/actuator away from the node.

The second approach is to use the scaled dynamic-dual of a two mode modal model (as in Figure 6(b)), capturing the dynamics of the first two modes of the cantilever. This is then connected to the same cantilever beam. The parameters of Num are the same as in the example of Figure 6 and those of Phy the same as those of the example of Figure 7.

Figure 8(a) shows the open (without the vibration absorber) and closed-loop (with the vibration absorber) frequency response magnitudes. This figure has been expanded to show the frequency responses close to the first–fourth resonances in Figure 8(c)–8(f), respectively. Near the first two resonances (Figure 8(c) and 8(d)), the magnitude of the closed-loop response (black line) is clearly smaller than the corresponding open-loop response (grey line) at the two resonant frequencies; these Figures are not the same as Figure 7(c) and 7(d) because there are only two control parameters. These were chosen as the gyrator gain $g^2 = 0.05$ and the damping of the cantilever beam model in Num as 2 per unit length.

Figure 7(a) shows the open (without the vibration absorber) and closed-loop (with the vibration absorber) frequency response magnitudes. This figure has been expanded to show the frequency responses close to the first–fourth resonances in Figure 7(c)–7(f), respectively. Near the first two resonances (Figure 7(c) and 7(d)), the magnitude of the closed-loop response (black line) is clearly smaller than the corresponding open-loop response (grey line) at the two resonant frequencies. The third resonance corresponds to a node at the sensor/actuator and the fourth is well damped anyway. Thus this controller design naturally applies control authority at the important resonances. Figure 7(b) shows the equivalent impulse response; as predicted by the frequency responses, the closed-loop impulse response decays more rapidly than the equivalent open-loop response. As noted above, the third resonance is not controlled using this approach. However, it could be controlled either by moving the sensor/actuator away from the node or by having a second sensor/actuator away from the node.
the controller is different; but the effect is similar. The third and fourth resonances are explicitly not controlled with this method; but, in this case, the effect is the same as that of the controller of the example of Figure 7. In particular Figure 8(b) shows that the closed-loop impulse response decays more rapidly than the equivalent open-loop response in a similar fashion to that of Figure 7(b). In this particular example, the performance of the two controllers is quite similar.

5. Experimental results
As indicated in Figure 9, the experiments were based on the Quanser (Apkarian, 1995) SRV02 rotational servo-motor and associated UPM-15-03-240 power and instrumentation module. The SRV02 was firmly clamped to a rigid bench and interfaced to an Intel CoreTM 2 Duo Processor (2.66 GHz) based computer via a National Instruments PCI-8024E analogue-digital conversion card and cable and the corresponding Quanser interface board.

In the experiment described here, the computer used the real-time Linux operating system RTAI together with the control-orientated software RTAI-Lab (Bucher & Balemi, 2006) running at a sampling frequency of 500 Hz. Using this software, the SRV02 rotational servo motor, rotational position sensor and associated power supply were controlled to give high-gain position control using a proportional and derivative (PD) controller. The servo angle was measured using a potentiometer and scaled within the computer to measure angular position in radians.
Figure 7. Cantilever beam with dual feedback. (a) Frequency responses $H(j\omega)$, (b) impulse responses $h(t)$, (c) $H(j\omega)$ – first resonance, (d) $H(j\omega)$ – second resonance, (e) $H(j\omega)$ – third resonance and (f) $H(j\omega)$ – fourth resonance.

5.1. Flexible joint

The Quanser cantilever beam experiment (Apkarian, 1995) has two parts that may be considered using the substructuring configuration shown in Figure 2(c). The physical component, $\text{Phy}$, consists of a rigid arm which is mounted to a platform via a pivot. This pivot exhibits a stiffness due to two linear springs mounted between the platform and the arm. A position disturbance is provided to the system via the rotational servo motor on which the platform is mounted (the pivot is directly above the motor). The vibration absorber, $\text{Num}$, has a torque input $F$. Because of the springs in $\text{Phy}$, this torque is proportional to the joint deflection angle $\theta$ (the arm rotation relative to the platform rotation), and so is generated from this measurement. The output of $\text{Num}$ is a rotational displacement $x$ which, along with the disturbance $x_0$, is imposed on $\text{Phy}$ using the servo motor by setting the servo motor PD controller demand to $x_0 - x$ (Figure 10).

The open-loop properties of the system were investigated by applying a square-wave reference signal with a period of 10 s to the servo and measuring both the servo angle $x_0$ and the joint angle $\theta$ for 5 periods. Because all of the signals are periodic, the methods of Pintelon & Schoukens (2001) were used to generate the frequency response of the system at the discrete frequencies corresponding to the periodic input. Figure 11 gives two measured frequency responses:

1. $+$ indicates the response from servo angle $x_0$ to joint angle $\theta$.
2. $\circ$ indicates the response from servo reference to $\theta$.

These responses match at low frequencies, but the gain of the second transfer function falls at the higher frequencies due to the limited servo bandwidth of about 10 Hz.
Figure 8. Cantilever with dual modal feedback. (a) Frequency responses $H(j\omega)$, (b) impulse responses $h(t)$, (c) $H(j\omega)$ — first resonance, (d) $H(j\omega)$ — second resonance, (e) $H(j\omega)$ — third resonance and (f) $H(j\omega)$ — fourth resonance.

With reference to Figures 2(c) and 10(a) the physical system $Phy$ relating input displacement to output force is of the form

$$s\tilde{p}(s, \theta_p) = \frac{g_0\omega_0^2}{s^2 + 2\xi_0\omega_0s + \omega_0^2}. \quad (30)$$

The parameters $\theta_p$ were fitted to the first measured frequency response with $\omega_0 = 15.1 \text{ rad} s^{-1}$ and $\xi_0 = 0.02$.

Following the methodology of Section 3.3 in the dual version of Figure 2(c), the feedback transfer function was chosen to be of the form:

$$\tilde{n}(s, \theta_n) = \frac{g_c}{s^2 + 2\xi_c\omega_0s + \omega_0^2}, \quad (31)$$

where $g_c = g^2g_0$ is a variable positive gain factor and the damping ratio $\xi_c = 0.3$.

The periodic input experimental method described above was used. Figure 12 shows the experimental frequency results for three values of $g_c$: $g_c = 0$, $g_c = 20$, and $g_c = 40$. $g_c = 0$ corresponds to Figure 11. The height of the resonant peak is reduced in the two non-zero cases and the peak splitting of Figure 5(c) is evident in Figure 12 for the highest gain of $g_c = 40$. Figure 13 shows the periodic data corresponding to the joint angle $\theta$ for the three gain values. The five consecutive periods have been superimposed to form the figures; the variability between periods is essentially high-frequency noise. As indicated by the frequency responses, the time responses show damping increasing with gain.
Figure 9. Experimental systems. The SRV02 servomotor module is in the bottom right-hand corner and the associated power module in the top left-hand corner. The computer display is at the top right and the computer interface board near the centre. The flexible joint module is shown mounted on the SRV02 and rotates about a vertical axis driven though the two springs. The cantilever beam module is shown unmounted and replaces the flexible joint module in the second set of experiments.

Figure 10. Rotational joint experiment. (a) With the components interpreted in a rotational sense and $r \rightarrow \infty$, $\text{Phy}$ represents the rotating arm with the attached springs. (b) $\text{Num}$ is the modified scaled dual of $\text{Phy}$. (a) $\text{Phy}$: the physical system and (b) $\text{Num}$: the dual physical system.
5.2. Cantilever beam

The flexible joint module was replaced by the cantilever beam module in Figure 9. A strain gauge measures the curvature at the root of the cantilever beam. In the same way as the joint potentiometer of Section 5.1 provided a voltage proportional to torque \( F \), the strain gauge sensor provides a voltage proportional to torque \( F \). The open-loop response was measured using the same methods. Two resonances and one anti-resonance appear in the measured frequency response and similarly to Section 5.1, this was fitted by a transfer function of the form:

\[
s\tilde{p}(s, \theta_p) = g_0 \left[ \frac{\kappa_1 s^2}{s^2 + 2\xi_1 \omega_1 s + \omega_1^2} + \frac{\kappa_2 s^2}{s^2 + 2\xi_2 \omega_2 s + \omega_2^2} \right]
\]

(32)

with \( \omega_1 = 23.25 \text{ rad s}^{-1} \), \( \omega_2 = 159.00 \text{ rad s}^{-1} \), \( \xi_1 = \xi_2 = 0.04 \), \( \kappa_1 = 0.36 \) and \( \kappa_2 = 1 - \kappa_1 = 0.64 \). Because of the 10 Hz servo bandwidth, the discrepancy between measured and fitted transfer function is large above 10 Hz.

Following the methodology of Section 4 a two-mode transfer function corresponding to Equation (33) was
The periodic input experimental method described above was used. Figure 15 shows the experimental frequency results for three values of $g_c$: $g_c = 0$, $g_c = 50$, and $g_c = 100$. $g_c = 0$ corresponds to Figure 14. The height of the first resonant peak is reduced in the two non-zero cases and the peak splitting of Figure 8(a) is evident in Figure 15 for both cases. The second resonance is largely unaffected; we attribute this to the limited actuator bandwidth. Figure 16 shows the periodic data corresponding to the measured strain voltage $\sigma$ for the three gain values. As with Figure 13, five consecutive periods have been superimposed to form the figures showing that the variability between periods is essentially high-frequency noise. Again, as indicated by the frequency responses, the time responses show damping increasing with gain.

6. Conclusion

The DDVA approach has been shown to provide a novel method to design vibration absorbers in the physical domain. In particular, the method is a natural generalisation of not only the classical single-degree of freedom vibration absorber of Den Hartog (1985, Section 3.3) but also of acceleration feedback (Preumont, 2002). Placing these two well-known design methods into the wider context of the DDVA of this paper has the following advantages: the methods immediately extend to multi degree of freedom systems and the implementation and theoretical results (including robustness to sensor/actuator dynamics) from substructuring can be directly applied.

The DDVA approach has been illustrated using numerical simulations of single mode and multi-mode systems and verified using two experimental systems: a rigid beam with a flexible joint and a flexible cantilever beam. Future work will apply the results to more complex dynamical systems including those with multiple sensor-actuator pairs.

The location of the sensor-actuator pairs has not been considered in this paper even though it certainly affects controllability and observability issues (Balas, 1978). Future work in this area will extend bond graph approaches (for example those of Marquis-Favre & Jardin (2011) and Gawthrop & Rizwi (2011)) to sensor/actuator placement in this context.

In principle, the method is equally applicable to the control nonlinear vibrations where dynamical dual of the nonlinear physical system provides the basis for a nonlinear controller. This is also an area for future work.

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Notes
1. As the word ‘dual’ has many meanings, the term dynamically dual is used for the specific meaning of this paper.
2. The bond directions have been changed for this paper to correspond to the usual sign convention for feedback control block diagrams.
3. ‘One-port’ refers to the single energy port associated with force and velocity.
4. In the sense that it consumes but does not produce energy.
5. Other models such as the Timoshenko model, as well as non-uniform beams, could similarly be handled using this approach.

References


### Appendix. Derivation of Equation (14)

Equation (14) can be derived directly from the bond graph of Figure 4(a). Letting \( F \) and \( v \) be the force and velocity at the component interface, letting \( F_{mr} \) be the force acting on the mass and damper and \( F_c \) the spring force, it follows that the components represented by \( I_m, C_k \) and \( R_r \) have equations:

\[
F_{mr} = m \frac{dv_m}{dt}, \quad (A1)
\]

\[
\frac{dF_c}{dt} = k v, \quad (A2)
\]

\[
v_r = \frac{1}{r} F_{mr}. \quad (A3)
\]

Taking Laplace transforms (with zero initial conditions) it follows that:

\[
v = v_r + v_m = \frac{1}{r} + \frac{1}{ms} F_{mr}
\]

\[
= \left( \frac{1}{r} + \frac{1}{ms} \right) (F - F_c)
\]

\[
= \frac{1}{r} + \frac{1}{ms} \left[ F - \frac{k}{s} v \right]
\]

\[
= \frac{ms + r}{mrs} \left[ F - \frac{k}{s} v \right]. \quad (A4)
\]

Collecting terms in Equation (A4) gives:

\[
\frac{k(ms + r) + mrs^2}{s(ms + r)} v = F. \quad (A5)
\]

Hence, rearranging Equation (A5):

\[
\frac{v}{F} = \frac{s(ms + r)}{mrs^2 + kms + kr}. \quad (A6)
\]

The right-hand side of Equation (A6) corresponds to the transfer function of Equation (14).