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Numerical Computations Concerning the GRH

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We describe two new algorithms for the efficient and rigorous computation of Dirichlet L-functions and their use to verify the Generalised Riemann Hypothesis for all such L-functions associated with primitive characters of modulus $q \leq 400,000$. We check to height $\max\left(\frac{10^8}{q}, \frac{A \cdot 10^7}{q} + 200\right)$ with $A = 7.5$ in the case of even characters and $A = 3.75$ for odd characters.

1 Introduction

For a given modulus $q \in \mathbb{Z}_{>0}$ we define the Dirichlet characters $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ axiomatically as follows:-

- $\chi(n) = 0$ iff $(n, q) \neq 1$,
- $\chi(mn) = \chi(m)\chi(n)$ and
- $\chi(n + q) = \chi(n)$.

There are $\varphi(q)$ distinct characters of modulus q , where φ is Euler’s totient function. The character $\chi(n) = 1$ for all n co-prime to q is known as the principal character. A character χ of modulus q is primitive if and only if for all d dividing q with $0 < d < q$ there exists an integer $a \equiv 1 \pmod{d}$ with $(a, q) = 1$ and $\chi(a) \neq 1$. [1]. Finally, we define the parity of a character by

$$a_\chi := \frac{1 - \chi(-1)}{2}.$$

The Dirichlet L-function of modulus q associated with a character χ is defined for $\Re s > 1$ by

$$L_\chi(s) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$$

and has analytic continuation to \mathbb{C} excepting (in the case of principal characters) a simple pole at $s = 1$.

Given ϵ_χ such that $|\epsilon_\chi| = 1$, we form the completed L-function via

$$\Lambda_\chi(t) := \epsilon_\chi \left(\frac{q}{\pi}\right)^{\frac{it}{2}} \Gamma\left(\frac{\frac{1}{2} + a_\chi + it}{2}\right) \exp\left(\frac{\pi t}{4}\right) L_\chi\left(\frac{1}{2} + it\right).$$

For suitably chosen ϵ_χ , Λ_χ is real valued and has the same zeros as $L_\chi\left(\frac{1}{2} + it\right)$. The exponential factor is introduced (for computational expedience) to counteract the decay of the gamma function as t increases.

In the case $q = 1$, we have only the principal character leading to a single L-function, namely Riemann's zeta function. Riemann's guess that all zeros of this function with real part in $[0, 1]$ lie on the $1/2$ line and are simple is the Riemann Hypothesis (RH). Extensive calculations have been undertaken to test RH to ever increasing heights, with Gourdon having checked the lowest 10^{13} zeros [6].

In contrast, the equivalent hypothesis for Dirichlet L-functions of primitive character, which we will refer to as the Generalised Riemann Hypothesis (GRH), has received less attention. The last significant rigorous computation was that by Rumely [16] who confirmed that the GRH holds for primitive L-functions modulus $q \leq 13$ to height 10,000 and various other moduli to height 2,500.* The largest modulus tested was $q = 432$ and in total about 10^7 zeros were examined. We note that Rumely went on to isolate these zeros with some precision and to generate statistics on their locations, but in terms of simply the number of zeros confirmed to lie on the $1/2$ line, there remained a factor of 10^6 in favour of zeta. If this weren't motivation enough, recent advances in the application of the Circle Method held out the tantalising prospect that ternary Goldbach might succumb to a combined numerical and analytic assault.

We will describe a computation using new algorithms and exploiting improvements in hardware in the 20 years since Rumely's paper that extend his result by about 6 orders of magnitude in terms of the number of zeros checked. Furthermore, the combination of moduli and heights checked is more than sufficient to support Helfgott's proof of ternary Goldbach [7][8].

2 Overview of Method

To verify GRH to height T for a particular Dirichlet L-function L_χ , we proceed as follows;

- Determine the number of zeros of $L_\chi(\sigma + it)$ with $\sigma \in [0, 1]$ and $t \in [0, T]$ using a variant of Turing's method which we describe in Section 5.
- Compute values of the associated completed L-function $\Lambda_\chi(t)$ for t going from 0 to T in steps of $1/A$ where A is a parameter to be determined but chosen to make $A \cdot T$ integral. We actually used two different methods depending on the size of the modulus of χ . Sections 6 and 7 refer.

*Rumley refers the Extended Riemann Hypothesis, rather than the GRH. This former term is now more often used to describe the hypothesis as related to zeros of the Dedekind zeta functions.

- Count the sign changes between $\Lambda_\chi(m/A)$ and $\Lambda_\chi((m+1)/A)$ for $m = 0 \dots A \cdot T - 1$, each of which indicates the presence of a zero of Λ_χ (by the Intermediate Value Theorem) and therefore of L_χ on the $1/2$ line.
- If the two counts match, we are done. If not, we presume that the rate at which we sampled Λ_χ was too coarse to isolate all its sign changes. Section 8 describes the method used to progressively increase our sampling rate when necessary.[†]

3 The Discrete Fourier Transform (DFT)

We will make extensive use of the DFT in what follows. We adopt the following (un-normalised) definition.

Definition 3.1. Given $N \in \mathbb{Z}_{>0}$ complex values denoted X_0 through X_{N-1} , the forward DFT results in N new values Y_0 through Y_{N-1} where

$$Y_m = \sum_{n=0}^{N-1} X_n e\left(\frac{-nm}{N}\right)$$

and as usual $e(x) := \exp(2\pi ix)$.

The backward or inverse DFT (iDFT) results from changing the sign in the complex exponential. Performing a forward then backward DFT (or vice versa) multiplies each datum by N . \square

As written, computing a DFT of length N would appear to have time complexity $\mathcal{O}(N^2)$. The ubiquity of the DFT stems from the existence of $\mathcal{O}(N \log N)$ algorithms, known collectively as the Fast Fourier Transforms (FFTs). For detailed descriptions of suitable algorithms, we refer the reader to, for example, [4]. However we note that, significantly for our purposes, this asymptotic complexity can be achieved for arbitrary (even prime) N . One such FFT, and the one we employ, is that due to Bluestein [2].

Throughout this paper, we will define \hat{f} , the (continuous) Fourier transform of a function f (when it exists), to be

$$\hat{f}(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \exp(-ixt) dt.$$

Under suitable conditions, the Fourier Inversion Theorem gives us

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(x) \exp(ixt) dx.$$

To make the transition from the discrete to the continuous, we use the following Theorem.

Theorem 3.2. Let f be a function in the Schwartz space with Fourier transform \hat{f} and $N = AB$ with $A, B > 0$. Define

$$\tilde{f}(n) := \sum_{l \in \mathbb{Z}} f\left(\frac{n}{A} + lB\right)$$

[†]We expect GRH to hold. If it fails in our domain of interest, our algorithm will not terminate.

and

$$\tilde{f}(m) := \sum_{l \in \mathbb{Z}} \hat{f}\left(\frac{2\pi m}{B} + 2\pi l A\right).$$

Then, up to a constant factor, $\tilde{f}(n)$ and $\tilde{f}(m)$ form a DFT pair of length N . □

Proof. By Poisson summation we have

$$\begin{aligned} \sum_{l \in \mathbb{Z}} f(t + lB) &= \frac{2\pi}{B} \sum_{l \in \mathbb{Z}} \hat{f}\left(\frac{2\pi l}{B}\right) e\left(\frac{lt}{B}\right) \\ \tilde{f}(n) &= \frac{2\pi}{B} \sum_{l \in \mathbb{Z}} \hat{f}\left(\frac{2\pi l}{B}\right) e\left(\frac{ln}{N}\right). \end{aligned}$$

We now write $l = l'N + m$ to get

$$\begin{aligned} \tilde{f}(n) &= \frac{2\pi}{B} \sum_{m=0}^{N-1} \sum_{l' \in \mathbb{Z}} \hat{f}\left(\frac{2\pi(l'N + m)}{B}\right) e\left(\frac{(l'N + m)n}{N}\right) \\ &= \frac{2\pi}{B} \sum_{m=0}^{N-1} e\left(\frac{mn}{N}\right) \tilde{f}(m). \end{aligned}$$

This is by definition an iDFT. ■

The utility of this theorem will be apparent when f and \hat{f} both decay quickly enough to allow $\tilde{f}(n)$ and $\tilde{f}(m)$ to be approximated by $f\left(\frac{n}{A}\right)$ and $\hat{f}\left(\frac{m}{B}\right)$ respectively.

4 Interval Arithmetic

Like Rumely, we chose to manage rounding and truncation errors throughout our computations using interval arithmetic. We refer the interested reader to the extensive literature on this subject (perhaps [12] is a good starting point) but we summarise our approach below.

Almost all real numbers cannot be represented by a floating point number of any given precision. Thus, whenever an operation is carried out on floating point numbers, unless we are very lucky, the answer will not be exactly representable. We typically attempt to round to the nearest real number that is exactly representable and thus incur a rounding error. Such errors will accumulate over time and, to quote Moore “it is often prohibitively difficult to tell in advance of a computation how many places must be carried to guarantee results of required accuracy.” [11].

Instead, we store our intermediate results as two exactly representable floating point numbers representing an interval that brackets the true result. The usual mathematical operators and functions are then abstracted to handle this new data type.

For high precision work (more than the 53 bits of IEEE double precision [9]) we use Revol and Rouillier’s MPFI package [15]. For computations where double precision will suffice, we use our own implementation based

on the work of Lambov [10] for $+$, $-$, \times , \div and sqrt . For \exp , \log , \sin , \cos and atan we use Muller and de Dinechin's "Correctly Rounded Mathematical Library" [13]. In both the high precision and double precision cases, we extend the real interval data type to the complexes in the obvious (and very probably sub-optimal) way, representing complex values as rectangles whose corners are exactly representable.

5 Turing's Method

Armed with the completed L-function, we have reduced the problem of locating simple zeros of L_χ on the half line to that of finding sign changes of Λ_χ . However, we still need a reference to confirm that all the expected zeros are accounted for. We use a variation on Turing's method [18], generalised by Booker.

Theorem 5.1 (Booker). Let $L(s)$ be an L-function given by an Euler product of degree r and absolutely convergent for $\Re s > 1$. Define

$$\begin{aligned}\Gamma_{\Re}(s) &:= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right), \\ \gamma(s) &:= \epsilon N^{\frac{1}{2}(s-\frac{1}{2})} \prod_{j=1}^r \Gamma_{\Re}(s + \mu_j), \\ \Lambda(s) &:= \gamma(s)L(s),\end{aligned}$$

where $|\epsilon| = 1$, $N \in \mathbb{Z}_{>0}$ and $\Re \mu_j \geq -\frac{1}{2}$ are chosen so that Λ satisfies the functional equation

$$\Lambda(s) = \overline{\Lambda(1 - \bar{s})}.$$

Now define

$$\begin{aligned}\Phi(t) &:= \frac{1}{\pi} \left[\arg \epsilon + \frac{\log N}{2} t - \frac{\log \pi}{2} \left(rt + \Im \sum_{j=1}^r \mu_j \right) \right. \\ &\quad \left. + \Im \sum_{j=1}^r \log \Gamma\left(\frac{1/2 + it + \mu_j}{2}\right) \right]\end{aligned}$$

and for t not the ordinate of a zero nor pole of Λ define

$$S(t) := \frac{1}{\pi} \int_{\infty}^{1/2} \frac{L'}{L}(\sigma + it) d\sigma.$$

Where t is the ordinate of a zero or pole, set $S(t) = \lim_{\delta \rightarrow 0^+} S(t + \delta)$ (i.e. S is upper semi-continuous). Finally, define

$$N(t) := \Phi(t) + S(t).$$

Then for $t_1 < t_2$, the net number of zeros with imaginary part in $[t_1, t_2)$ counting multiplicity is $N(t_2) - N(t_1)$. \square

Proof. See §4 of [3]. ■

Theorem 5.2. Given $T, h > 0$ such that neither T nor $T + h$ is the imaginary part of a zero of $L_\chi(s)$, let $N_\chi(T)$ be the number of zeros, counted with multiplicity, of $L_\chi(s)$ with $|\Im(s)| \leq T$ and $\Re(s) \in (0, 1)$. Let $\check{N}_{T,\chi}(t)$ count the zeros of $L_\chi(s)$ with $\Im(s) \in [T, t)$, starting at 0 at T and increasing by 1 at every zero.

Now for t not the ordinate of a zero of L_χ , define $S_\chi(t)$ by

$$S_\chi(t) := \frac{1}{\pi} \Im \int_{\infty}^{\frac{1}{2}} \frac{L'_\chi}{L_\chi}(\sigma + it) d\sigma$$

and take $S_\chi(t)$ to be upper semi-continuous. Then we have

$$N_\chi(T) = \frac{1}{h\pi} \left[2h + \frac{2hT + h^2}{2} \log\left(\frac{q}{\pi}\right) + 2 \int_T^{T+h} \Im \log \Gamma\left(\frac{1/2 + a_\chi + it}{2}\right) dt \right. \\ \left. - \int_T^{T+h} \check{N}_{T,\chi}(t) dt - \int_T^{T+h} \check{N}_{T,\bar{\chi}}(t) dt + \int_T^{T+h} S_\chi(t) dt + \int_T^{T+h} S_{\bar{\chi}}(t) dt \right]. \quad \square$$

Proof. This is Theorem 5.1 specialised to Dirichlet L-functions. In the terminology of that Theorem, we have $N = q$, $r = 1$ and $\mu_1 = a_\chi$. We treat conjugate characters in pairs to avoid problems with the arbitrary choice of ϵ_χ and to allow for the possibility that $S_\chi(0)$ isn't small. Finally, we integrate both sides from T to $T + h$. ■

This leaves us with the problem of bounding $\int_T^{T+h} S_\chi(t) dt$.

Theorem 5.3. (Rumely). For $T > 50$ and $h > 0$

$$\left| \int_T^{T+h} S_\chi(t) dt \right| \leq 1.8397 + 0.1242 \log\left(\frac{q(T+h)}{2\pi}\right). \quad \square$$

Proof. Theorem 2 of [16]. ■

Trudgian considered this problem in [17]. Specifically, applying Theorem 3.8 of that paper with $c = 1.1$ and $d = 0.8$ we derive revised constants optimised for qT in the region of 10^8 . We have

Theorem 5.4. (Trudgian). For $T > 50$ and $h > 0$

$$\left| \int_T^{T+h} S_\chi(t) dt \right| \leq 2.17618 + 0.0679956 \log\left(\frac{q(T+h)}{2\pi}\right). \quad \square$$

The two bounds agree near $qt = 2,501$ and at $qt = 10^8$ Trudgian's is better by a little more than 0.5.

6 An Algorithm for Large q

For “large” moduli, we compute the values of $L_\chi(s)$ simultaneously for all characters of a given modulus by expressing the calculations as a Discrete Fourier Transform. Specifically, we appeal to the following lemma.

Lemma 6.1. For $q \in \mathbb{Z} \geq 3$ and given $\varphi(q)$ complex values $a(n)$ for $n \in [1, q-1]$ and $(n, q) \neq 0$, we can compute

$$\sum_{n=1}^{q-1} a(n)\chi(n)$$

for the $\varphi(q)$ characters χ in $\mathcal{O}(\varphi(q) \log(q))$ time and $\mathcal{O}(\varphi(q))$ space. \square

Proof. Let $U(R)$ be the group of units of the ring R . For $q \in \mathbb{Z}_{>0}$ with the prime decomposition $q = 2^\alpha \prod_{i=1}^m p_i^{\alpha_i}$.

We consider four cases;

1. $\alpha = 0$ (q is odd) then by the Chinese Remainder Theorem (CRT) we have the constructive, canonical group isomorphism

$$U(\mathbb{Z}/q\mathbb{Z}) \cong \prod_{i=1}^m U(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}).$$

Each of these groups is cyclic so given a primitive root for each $p_i^{\alpha_i}$ we have our construction. Thus this case reduces to performing $\varphi(q)/\varphi(p_i^{\alpha_i})$ length $\varphi(p_i^{\alpha_i})$ DFTs for $i = 1 \dots m$.

2. $\alpha = 1$ then by the CRT we have the constructive group isomorphism

$$U(\mathbb{Z}/q\mathbb{Z}) \cong U(\mathbb{Z}/2p_1^{\alpha_1}\mathbb{Z}) \prod_{i=2}^m U(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}).$$

Each of these groups is cyclic so given a primitive root for $2p_1^{\alpha_1}$ and each $p_i^{\alpha_i}$ ($i > 1$) we have our construction. Thus this case reduces to performing $\varphi(q)/\varphi(2p_1^{\alpha_1})$ length $\varphi(2p_1^{\alpha_1})$ DFTs followed by $\varphi(q)/\varphi(p_i^{\alpha_i})$ length $\varphi(p_i^{\alpha_i})$ DFTs for $i = 2 \dots m$.

3. $\alpha = 2$ then by the CRT we have the constructive, canonical group isomorphism

$$U(\mathbb{Z}/q\mathbb{Z}) \cong U(\mathbb{Z}/4\mathbb{Z}) \prod_{i=1}^m U(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}).$$

Each of these groups is cyclic so given a primitive root for each $p_i^{\alpha_i}$ ($i > 1$) we have our construction. Thus this case reduces to performing $\varphi(q)/2$ length 2 DFTs followed by $\varphi(q)/\varphi(p_i^{\alpha_i})$ length $\varphi(p_i^{\alpha_i})$ DFTs for $i = 1 \dots m$.

4. $\alpha > 2$ then by the CRT we have the constructive, canonical group isomorphism

$$U(\mathbb{Z}/q\mathbb{Z}) \cong U(\mathbb{Z}/2^\alpha\mathbb{Z}) \prod_{i=1}^m U(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}).$$

Now $U(\mathbb{Z}/2^\alpha\mathbb{Z})$ is the product of a cyclic group of order 2 and a cyclic group of order $2^{\alpha-2}$ with pseudo primitive roots -1 and 5 respectively. The remaining groups (if there are any) are cyclic so given a primitive root for each $p_i^{\alpha_i}$ ($i > 1$) we have our construction. Thus this case reduces to performing $\varphi(q)/2$ length 2 DFTs, $\varphi(q)/2^{\alpha-2}$ length $2^{\alpha-2}$ DFTs followed by $\varphi(q)/\varphi(p_i^{\alpha_i})$ length $\varphi(p_i^{\alpha_i})$ DFTs for $i = 1 \dots m$.

In each case, given the ability to perform an arbitrary length n DFT in time $\mathcal{O}(n \log n)$, we have the claimed overall complexity. \blacksquare

We seek to apply Lemma 6.1 by way of the Hurwitz zeta function, defined for $\Re s > 1$ and $\alpha \in (0, 1]$ by

$$\zeta(s, \alpha) := \sum_{n=0}^{\infty} (n + \alpha)^{-s}.$$

This function has analytic continuation to \mathbb{C} with the exception of a simple pole at $s = 1$ and except at this pole it can be used to express any Dirichlet L-function of modulus q via

$$L_\chi(s) = q^{-s} \sum_{a=1}^q \chi(a) \zeta\left(s, \frac{a}{q}\right)$$

(see § 12 of [1]).

Thus, for a given q and s , if we can supply the $\varphi(q)$ values of $\zeta\left(s, \frac{a}{q}\right)$ for $a \in [1, q-1]$ with $(a, q) = 1$, we can apply Lemma 6.1 to compute each $L_\chi(s)$ in, on average, time $\log q$.

6.1 Computing $\zeta(1/2 + it, a/q)$

For a given $t \in \mathbb{R}_{\geq 0}$ and $q \geq 3$, we need to be able to compute $\zeta(1/2 + it, a/q)$ for $a \in [1, q]$ with $(a, q) = 1$. We proceed by computing, for each t , a lattice of D rows and N columns where the entry in the r 'th row and c 'th column are $\zeta(1/2 + it + c, r/D)$ (r running $1 \dots D$ and c $0 \dots N$). We chose $N = 15$ and $D = 2,048$ to achieve the necessary precision.

We use the following lemma both to initially compute the lattice (once, in high precision using MPFI) and to compute the required values for the DFT from that lattice (many times, using double precision intervals).

Lemma 6.2. For $s \notin \mathbb{Z}_{\leq 0}$, $\alpha \in (0, 1]$ and $|\delta| < \alpha$

$$\zeta(s, \alpha + \delta) = \sum_{k=0}^{\infty} \frac{(-\delta)^k \zeta(s + k, \alpha) \prod_{j=0}^{k-1} (s + j)}{k!}. \quad \square$$

Proof. Starting with $\Re s > 1$ and differentiating term by term we have

$$\zeta^{(k)}(s, \alpha) = \sum_{n=0}^{\infty} (-1)^k s(s+1)(s+2) \dots (s+k-1)(n+\alpha)^{-s-k}$$

and the result follows for $\Re s > 1$ by Taylor's Theorem. The Taylor expansion also gives us the analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. ■

In practice, it is better to work with

$$\zeta_M(s, \alpha) = \zeta(s, \alpha) - \sum_{n=0}^M (n + \alpha)^{-s}$$

for some $M \in \mathbb{Z}_{>0}$ and to recover $\zeta(s, \alpha)$ by adding back the missing terms.

7 An Algorithm for Small q

The algorithm described above starts to become unwieldy as t , the height up the critical line, increases. Each new t requires its own pre-computed lattice of $\zeta(1/2 + it + c, r/D)$ and the cost of producing this lattice is amortised over less and less q . In [3], Booker describes a rigorous algorithm for computing L-functions. What follows is that algorithm specialised to Dirichlet L-functions.

For $\eta \in (-1, 1)$ and even primitive characters χ define

$$F_e(t, \chi) := \epsilon_\chi q^{\frac{it}{2}} \pi^{-\frac{1/2+it}{2}} \Gamma\left(\frac{1/2+it}{2}\right) \exp\left(\frac{\pi\eta t}{4}\right) L_\chi\left(\frac{1}{2} + it\right) \text{ and}$$

$$\hat{F}_e(x, \chi) := \frac{1}{2\pi} \int_{-\infty}^{\infty} F_e(t, \chi) e^{-ixt} dt.$$

For odd primitive characters χ define

$$F_o(t, \chi) := \epsilon_\chi q^{\frac{it}{2}} \pi^{-\frac{3/2+it}{2}} \Gamma\left(\frac{3/2+it}{2}\right) \exp\left(\frac{\pi\eta t}{4}\right) L_\chi\left(\frac{1}{2} + it\right) \text{ and}$$

$$\hat{F}_o(x, \chi) := \frac{1}{2\pi} \int_{-\infty}^{\infty} F_o(t, \chi) e^{-ixt} dt.$$

We chose the parameter η to control the decay of the gamma factor as t increases.

We now choose $A, B > 0$ with $N = AB \in 2^{\mathbb{Z}_{>0}}$ and define

$$\tilde{F}_e(n, \chi) := \sum_{k \in \mathbb{Z}} \hat{F}_e\left(\frac{2\pi n}{B} + 2\pi k A, \chi\right)$$

and

$$\tilde{F}_o(n, \chi) := \sum_{k \in \mathbb{Z}} \hat{F}_o\left(\frac{2\pi n}{B} + 2\pi k A, \chi\right).$$

Similarly, define

$$\tilde{F}_e(m, \chi) := \sum_{k \in \mathbb{Z}} F_e \left(\frac{m}{A} + kB, \chi \right)$$

and

$$\tilde{F}_o(m, \chi) := \sum_{k \in \mathbb{Z}} F_o \left(\frac{m}{A} + kB, \chi \right).$$

In outline, the method is

1. Compute $\hat{F}_e \left(\frac{2\pi n}{B} \right)$ or $\hat{F}_o \left(\frac{2\pi n}{B} \right)$ for $n = 0 \dots N - 1$.
2. Use these values as an approximation to $\tilde{F}_e(n, \chi)$ or $\tilde{F}_o(n, \chi)$ respectively.
3. Appealing to Theorem 3.2, perform a DFT to yield $\tilde{F}_e(m, \chi)$ or $\tilde{F}_o(m, \chi)$ respectively.
4. Use $\tilde{F}_e(m, \chi)$ or $\tilde{F}_o(m, \chi)$ as an approximation to $F_e \left(\frac{m}{A}, \chi \right)$ or $F_o \left(\frac{m}{A}, \chi \right)$ respectively.

We now make the above outline rigorous.

7.1 Computing $\hat{F}_e(t)$ and $\hat{F}_o(t)$

Lemma 7.1. Let $x \in \mathbb{R}$, $\eta \in (-1, 1)$ and $u(x) := \frac{\pi\eta i}{4} + x$. Then we have

$$\hat{F}_e(x, \chi) = \frac{2\epsilon_\chi \exp\left(\frac{u(x)}{2}\right)}{q^{\frac{1}{4}}} \sum_{n=1}^{\infty} \chi(n) \exp\left(-\frac{\pi n^2 \exp(2u(x))}{q}\right). \quad \square$$

Proof. Writing $s = 1/2 + it$ we get

$$\begin{aligned} \hat{F}_e(x, \chi) &= \frac{\epsilon_\chi}{2\pi i} \int_{\Re(s)=\frac{1}{2}} q^{\frac{s-1/2}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \exp\left(\frac{-(\pi\eta i + 4x)(s-1/2)}{4}\right) L_\chi(s) ds \\ &= \frac{\epsilon_\chi}{2\pi i} \int_{\Re(s)=2} q^{\frac{s-1/2}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \exp(-u(x)(s-1/2)) L_\chi(s) ds \\ &= \frac{\epsilon_\chi \exp\left(\frac{u(x)}{2}\right)}{q^{\frac{1}{4}}} \frac{1}{2\pi i} \int_{\Re(s)=2} \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \exp(2u(x))^{-\frac{s}{2}} \sum_{n=1}^{\infty} \chi(n) n^{-s} ds \\ &= \frac{\epsilon_\chi \exp\left(\frac{u(x)}{2}\right)}{q^{\frac{1}{4}}} \sum_{n=1}^{\infty} \chi(n) \frac{1}{2\pi i} \int_{\Re(s)=2} \left(\frac{\pi n^2}{q}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \exp(2u(x))^{-\frac{s}{2}} ds \\ &= \frac{2\epsilon_\chi \exp\left(\frac{u(x)}{2}\right)}{q^{\frac{1}{4}}} \sum_{n=1}^{\infty} \chi(n) \exp\left(-\frac{\pi n^2 \exp(2u(x))}{q}\right) \end{aligned}$$

as required. ■

We can rigorously bound the error in truncating the sum either by reference to Lemma 5.4 of [3] or by majorising the missing terms with the obvious geometric series.

Lemma 7.2. Let x, η and $u(x)$ be as defined in Lemma 7.1. Then we have

$$\hat{F}_o(x, \chi) = \frac{2\epsilon_\chi \exp\left(\frac{3u(x)}{2}\right)}{q^{\frac{3}{4}}} \sum_{n=1}^{\infty} n\chi(n) \exp\left(-\frac{\pi n^2 \exp(2u(x))}{q}\right). \quad \square$$

Proof. The proof follows the same lines as Lemma 7.1. ■

7.2 Approximating \tilde{F}_e and \tilde{F}_o with \hat{F}_e and \hat{F}_o

We intend to chose our parameters to allow us to use \hat{F}_e and \hat{F}_o as approximations to \tilde{F}_e and \tilde{F}_o respectively.

We therefore need to bound the error introduced and we start with two lemmas.

Lemma 7.3. For $t \in \mathbb{R}$ we have

$$\left|L_\chi\left(\frac{1}{2} + it\right)\right| \leq \zeta\left(\frac{9}{8}\right) \left(\frac{q}{2\pi}\right)^{5/16} \left(\frac{3}{2} + |t|\right)^{5/16}. \quad \square$$

Proof. We evaluate Rademacher's bound [14]

$$|L_\chi(s)| \leq \zeta(1 + \nu) \left(\frac{q|1 + s|}{2\pi}\right)^{\frac{1 + \nu - \Re(s)}{2}}$$

with $\nu = 1/8$ and $s = 1/2 + it$. ■

Lemma 7.4. (Booker) Let $\eta \in (-1, 1)$, $\delta = \frac{\pi}{2}(1 - |\eta|)$ and $X(x) = \pi\delta e^{-\delta}x > 1$. Then

$$\left|\sum_{k=0}^{\infty} \hat{F}_e(x + 2\pi kA)\right| \leq \frac{4 \exp\left(\frac{x}{2} - X(x)\right) \left(1 + \frac{1}{2X(x)}\right)}{\delta^{\frac{1}{2}} q^{\frac{1}{4}} (1 - e^{-\pi A})}$$

and

$$\left|\sum_{k=0}^{\infty} \hat{F}_o(x + 2\pi kA)\right| \leq \frac{4 \exp\left(\frac{3x}{2} - X(x)\right) \left(1 + \frac{1}{2X(x)}\right)^{\frac{3}{2}}}{\delta^{\frac{1}{2}} q^{\frac{3}{4}} (1 - e^{-\pi A})}. \quad \square$$

Proof. This is Lemma 5.6 of [3] specialised to Dirichlet L-functions. ■

We can now proceed to the necessary bounds.

Lemma 7.5. Let $A \geq \frac{1}{2\pi}$, $B > 0$, $w_1 = \frac{2\pi n}{B} + 2\pi A$, $w_2 = -\frac{2\pi n}{B} + 2\pi A$, with $X(x)$ and δ as defined in Lemma 7.4 and $X(w_1), X(w_2) > 1$. Then

$$\left|\tilde{F}_e(n, \chi) - \hat{F}_e\left(\frac{2\pi n}{B}, \chi\right)\right| \leq \frac{4 \left(\exp\left(\frac{w_1}{2} - X(w_1)\right) \left(1 + \frac{1}{2X(w_1)}\right) + \exp\left(\frac{w_2}{2} - X(w_2)\right) \left(1 + \frac{1}{2X(w_2)}\right)\right)}{q^{1/4} \delta^{1/2} (1 - e^{-\pi A})}$$

and

$$\left| \tilde{F}_o(n, \chi) - \hat{F}_o\left(\frac{2\pi n}{B}, \chi\right) \right| \leq \frac{4 \left(\exp\left(\frac{3w_1}{2} - X(w_1)\right) \left(1 + \frac{1}{2X(w_1)}\right)^{\frac{3}{2}} + \exp\left(\frac{3w_2}{2} - X(w_2)\right) \left(1 + \frac{1}{2X(w_2)}\right)^{\frac{3}{2}} \right)}{q^{3/4} \delta^{1/2} (1 - e^{-\pi A})}. \quad \square$$

Proof. We apply Lemma 7.4 with $x = \frac{2\pi n}{B} \pm 2\pi A$. ■

Lemma 7.6. Given $t \in \mathbb{R}$ and $B > 0$, we define

$$E_e(t) := \zeta\left(\frac{9}{8}\right) \pi^{-\frac{1}{4}} \left| \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \right| e^{\frac{\pi}{4}\eta t} \left(\frac{q}{2\pi} \left| \frac{3}{2} + t \right| \right)^{\frac{5}{16}},$$

$$\beta_e(t) := \frac{\pi}{4} - \frac{1}{2} \arctan\left(\frac{1}{2|t|}\right) - \frac{4}{\pi^2 |t^2 - \frac{1}{4}|},$$

$$E_o(t) := \zeta\left(\frac{9}{8}\right) \pi^{-\frac{3}{4}} \left| \Gamma\left(\frac{3}{4} + \frac{it}{2}\right) \right| e^{\frac{\pi}{4}\eta t} \left(\frac{q}{2\pi} \left| \frac{3}{2} + t \right| \right)^{\frac{5}{16}}$$

and

$$\beta_o(t) := \frac{\pi}{4} - \frac{3}{2} \arctan\left(\frac{1}{2|t|}\right) - \frac{4}{\pi^2 |t^2 - \frac{9}{4}|}.$$

Then for $\beta_{e,o}\left(\frac{m}{A} + B\right) > \frac{\pi}{4}\eta$ and $\beta_{e,o}\left(\frac{m}{A} - B\right) > -\frac{\pi}{4}\eta$ we have

$$\left| \tilde{F}_e(m, \chi) - F_e\left(\frac{m}{A}, \chi\right) \right| \leq \frac{E_e\left(\frac{m}{A} + B\right)}{1 - \exp(-B(\beta_e(m/A + B) - \frac{\pi}{4}\eta))} + \frac{E_e\left(\frac{m}{A} - B\right)}{1 - \exp(-B(\beta_e(m/A - B) + \frac{\pi}{4}\eta))}$$

and

$$\left| \tilde{F}_o(m, \chi) - F_o\left(\frac{m}{A}, \chi\right) \right| \leq \frac{E_o\left(\frac{m}{A} + B\right)}{1 - \exp(-B(\beta_o(m/A + B) - \frac{\pi}{4}\eta))} + \frac{E_o\left(\frac{m}{A} - B\right)}{1 - \exp(-B(\beta_o(m/A - B) + \frac{\pi}{4}\eta))}. \quad \square$$

Proof. We apply Lemma 5.7 (i) of [3] with $t = \frac{m}{A} + B$ and 5.7 (ii) with $t = \frac{m}{A} - B$, replacing the bound for $L_\chi(s)$ with our Lemma 7.3. ■

We note here that the condition on $\beta_{e,o}(t)$ will fail when t is small, i.e. when $\frac{m}{A} \approx B$. However, this only happens for m approaching AB , by which point the loss of precision through other factors has rendered these values useless for computational purposes anyway.

8 Rigorous Up-sampling

The algorithms described in Sections 6 and 7 both result in a lattice of values of $\Lambda_\chi(t)$. The sample rate used (5/64) will be insufficient to resolve all the zeros, so we employ a rigorous up-sampling technique based on theorems of Whittaker-Shannon and Weiss.

Theorem 8.1. (Whittaker-Shannon Sampling Theorem) Let $f(t)$ be a continuous, real valued function with Fourier Transform $\hat{f}(x)$ such that $\hat{f}(x) = 0$ for $|x| > 2\pi B > 0$ (i.e. $f(t)$ is band-limited with bandwidth $2\pi B$). Also, define

$$\text{sinc}(x) := \frac{\sin(x)}{x}.$$

Then

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2B}\right) \text{sinc}\left(2B\pi\left(\frac{n}{2B} - t\right)\right),$$

when this sum converges. □

Proof. See [19]. ■

To apply Theorem 8.1 rigorously, we need to examine two sources of error

- the error introduced by truncating the sum and
- the error introduced if the function is only approximately band-limited.

The former will be dealt with on a case by case basis. The latter, referred to as aliasing in signal processing circles, is the subject of a theorem due to Weiss.

Theorem 8.2. (Weiss) Let $f(t)$ be a real valued function with Fourier Transform $\hat{f}(x)$ such that

1. $\int_{-\infty}^{\infty} |\hat{f}(x)| dx < \infty$
2. $\hat{f}(x)$ is of bounded variation on \mathbb{R}
3. when \hat{f} has a jump discontinuity at x , then $\hat{f}(x) = \lim_{\epsilon \rightarrow 0^+} \frac{\hat{f}(x-\epsilon) + \hat{f}(x+\epsilon)}{2}$.

Then

$$\left| f(t) - \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2B}\right) \text{sinc}\left(2B\pi\left(t - \frac{n}{2B}\right)\right) \right| \leq 4 \int_{2\pi B}^{\infty} |\hat{f}(x)| dx. \quad \square$$

Proof. See for example [5]. ■

For $t_0 \in \mathbb{R}$ and $h > 0$ define $W : \mathbb{R} \rightarrow \mathbb{R}$ by

$$W(t, \chi) := \Lambda_\chi(t) \exp\left(\frac{-(t - t_0)^2}{2h^2}\right)$$

so $W(t_0, \chi) = \Lambda_\chi(t)$.

We aim to estimate $W(t_0, \chi)$ from our samples using Theorems 8.1 (Whittaker-Shannon) and 8.2. The following lemmas provide the necessary rigorous bounds.

Lemma 8.3. For $a_\chi \in \{0, 1\}$

$$\left| \Gamma\left(\frac{\frac{1}{2} + it + a_\chi}{2}\right) \right| e^{\frac{\pi t}{4}} \leq \max\left(2^{1/4} \sqrt{\pi} \left(\frac{3}{2} + \max(t, 0)\right)^{\frac{1}{4}} \exp\left(\frac{1}{6}\right), \sqrt{2\pi} \exp\left(\frac{\pi}{8} + \frac{1}{4}\right)\right). \quad \square$$

Proof. We use Stirling's approximation separately for $a_\chi = 0$ and $a_\chi = 1$. ■

Lemma 8.4. Define I_χ by

$$I_\chi := \frac{2}{\pi} \int_{2\pi B}^{\infty} \left| \int_{-\infty}^{\infty} W(t, \chi) \exp(-ixt) dt \right| dx.$$

Then, writing M in place of $\frac{5}{2} - a_\chi$ we have

$$I_\chi \leq \frac{2 \left(\frac{q}{\pi}\right)^{\frac{M}{2}} \zeta(M + 1/2) \exp\left(\frac{M^2}{2h^2} - 2\pi BM\right) P(t_0, h)}{\pi M}$$

where

$$P(t_0, h) = \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{3 + it}{2}\right) \right| \exp\left(\frac{\pi t}{4}\right) \exp\left(-\frac{(t - t_0)^2}{2h^2}\right) dt. \quad \square$$

Proof. Writing $s = 1/2 + it$ we get

$$I_\chi \leq \frac{2}{\pi} \int_{2\pi B}^{\infty} \left| \int_{\Re(s)=1/2}^{\infty} \left(\frac{q}{\pi}\right)^{\frac{s-1/2}{2}} \Gamma\left(\frac{s + a_\chi}{2}\right) \exp\left(\frac{\pi i(1/2 - s)}{4}\right) L_\chi(s) \exp((1/2 - s)x) \exp\left(\frac{-(i(1/2 - s) - t_0)^2}{2h^2}\right) ds \right| dx.$$

We now shift the contour of integration to the right so that $\Re(s) = \sigma = 3 - a_\chi$ and write $s = M + 1/2 + it$ to get

$$I_\chi \leq \frac{2}{\pi} \int_{2\pi B}^{\infty} \int_{-\infty}^{\infty} \left| \left(\frac{q}{\pi}\right)^{\frac{M}{2}} \Gamma\left(\frac{3 + it}{2}\right) \exp\left(\frac{\pi t}{4}\right) \zeta(M + 1/2) \exp(-Mx) \exp\left(\frac{M^2 - (t - t_0)^2}{2h^2}\right) \right| dt dx.$$

Integrating with respect to t gives us

$$I_\chi \leq \frac{2}{\pi} \left(\frac{q}{\pi}\right)^{\frac{M}{2}} \zeta(M + 1/2) \exp\left(\frac{M^2}{2h^2}\right) P(t_0, h) \int_{2\pi B}^{\infty} \exp(-Mx) dx$$

and the result follows after integrating with respect to x . ■

Lemma 8.5. Let $t_0 \geq 0$. Then

$$P(t_0, h) \leq h\pi \left(t_0 + \frac{h}{\sqrt{2\pi}} + 1 + \frac{1}{2\sqrt{2}} \right). \quad \square$$

Proof. We have

$$\begin{aligned} P(t_0, h) &\leq \int_0^\infty \left| \Gamma\left(\frac{3+it}{2}\right) \right| \exp\left(\frac{\pi t}{4}\right) \exp\left(\frac{-(t-t_0)^2}{2h^2}\right) dt \\ &\quad + \int_{-\infty}^0 \left| \Gamma\left(\frac{3+it}{2}\right) \right| \exp\left(\frac{\pi t}{4}\right) \exp\left(\frac{-(t-t_0)^2}{2h^2}\right) dt \\ &\leq \int_0^\infty \frac{(1+t)}{2} \left| \Gamma\left(\frac{1+it}{2}\right) \right| \exp\left(\frac{\pi t}{4}\right) \exp\left(\frac{-(t-t_0)^2}{2h^2}\right) dt \\ &\quad + \Gamma\left(\frac{3}{2}\right) \frac{h\sqrt{2\pi}}{2} \left(1 - \operatorname{erf}\left(\frac{\sqrt{2}t_0}{2}\right)\right) \\ &\leq \int_0^\infty \frac{(1+t)}{2} \sqrt{\frac{\pi}{\cosh(\pi t/2)}} \exp\left(\frac{\pi t}{4}\right) \exp\left(\frac{-(t-t_0)^2}{2h^2}\right) dt \\ &\quad + \Gamma\left(\frac{3}{2}\right) \frac{h\sqrt{2\pi}}{2} \\ &\leq \int_0^\infty \frac{(1+t)}{2} \sqrt{2\pi} \exp\left(\frac{-(t-t_0)^2}{2h^2}\right) dt + \frac{h\pi\sqrt{2}}{4} \\ &\leq h\pi \left(\frac{h}{\sqrt{2\pi}} + t_0 + 1 \right) + \frac{h\pi\sqrt{2}}{4} \end{aligned}$$

as claimed. ■

Lemma 8.6. Let $h, B > 0$, $t_0 = \frac{n_0}{2B}$ for some $n_0 \in \mathbb{Z}_{>0}$ and $N \in \mathbb{Z}_{>0}$. Now define

$$G(n) := \frac{\left(\frac{3}{2} + t_0 + \frac{N+n}{2B}\right)^{9/16} \exp\left(\frac{-(N+n)^2}{8B^2h^2}\right)}{\pi(N+n)}.$$

Then

$$\sum_{n \geq 2Bt_0 + N} \left(\frac{3}{2} + \frac{n}{2B}\right)^{9/16} \exp\left(\frac{-\left(\frac{n}{2B} - t_0\right)^2}{2h^2}\right) \operatorname{sinc}\left(2B\pi\left(\frac{n}{2B} - t_0\right)\right) \leq \frac{G(0)}{1 - G(1)/G(0)}. \quad \square$$

Proof. $G(n)$ is at least as large as the corresponding term in the sum and the ratio $G(n+1)/G(n)$ is a decreasing function of n so the result follows as the sum of a geometric series. ■

We can now combine Lemmas 7.3, 8.3 and 8.6.

Lemma 8.7. Define

$$E := \sum_{|n| \geq N} W\left(\frac{n}{2B}\right) \operatorname{sinc}\left(2B\pi\left(\frac{n}{2B} - t_0\right)\right).$$

Then for large enough t_0 we have

$$|E| \leq \sqrt{\pi} \zeta\left(\frac{9}{8}\right) \exp(1/6) 2^{5/4} \left(\frac{q}{2\pi}\right)^{5/16} \frac{G(0)}{1 - G(1)/G(0)}. \quad \square$$

9 Results

Both algorithms parallelise trivially and were run on various clusters in the UK and France. We ensured that every modulus was checked at some point on a system benefiting from ECC memory using the small q algorithm up to $q = 10,000$ or so, and the large q algorithm beyond that.

We chose $A = 64/5$ so that we moved up the critical line in steps of $\frac{5}{64}$. This represents a sampling rate of about 5 times the expected zero density. For upsampling, we set $h = 7/32$ and used 20 samples either side of t_0 . This gave us an up-sampling error $< 8.6 \cdot 10^{-8}$. We routinely up-sampled by a factor of 8 and then if necessary by 32, 128 and ultimately 512. At this point, about 0.0003% of the L-functions remained due to one or more of the following issues:-

- The sign of $\Lambda_\chi(1/2)$ could not be determined. This was resolved using a double precision interval implementation of Euler-MacLaurin.
- The sign of Λ_χ was positive, became indeterminate and then became positive again (or negative, indeterminate, negative). Since a failure to cross the x axis here would, on its own, be enough to refute GRH, we fully expected to find that the indeterminate region was actually hiding a pair of zeros. In every case, using an interval arithmetic version of Euler MacLaurin (first at double precision, but occasionally resorting to MPFI at 100 bits) located the expected sign changes.
- The sign of Λ_χ was positive, indeterminate and then negative (or vice versa). Rather than hiding a single sign change, closer inspection revealed three sign changes in the indeterminate region.
- Occasionally, the estimate for the number of zeros to locate computed via Turing's method did not bracket an integer. This was caused by zeros being missed in the region used to compute the Turing estimate itself and these were resolved by shifting the region or locating the missing zeros using high precision.

In all, the computation consumed approximately 400,000 core hours.[‡] We checked all the 29,565,923,837 Dirichlet L-functions with primitive modulus $q \leq 400,000$, isolating approximately $3.8 \cdot 10^{13}$ zeros (not counting those used in Turing's method). Specifically, we have;

Theorem 9.1. GRH holds for Dirichlet L-functions of primitive character modulus $q \leq 400,000$ and to height $T = \max\left(\frac{10^8}{q}, \frac{7.5 \cdot 10^7}{q} + 200\right)$ for even q and to height $T = \max\left(\frac{10^8}{q}, \frac{3.75 \cdot 10^7}{q} + 200\right)$ for odd q . □

In addition, we explored the central point of the 739,151,526,102 primitive characters with $q \leq 2,000,000$ using the large q algorithm. In 438,152 cases, the computation returned a value for the completed L-function as

[‡]The computing resources used were Intel/AMD based and equipped with the SSE2 instruction set. Except for small q , where the lengths of the FFTs involved became the limiting factor, we were able to exploit all of the cores available to us on multi-core systems.

a double precision interval that straddled zero. Recomputing these points, again using double precision intervals but this time via Euler-MacLaurin, resolved all but 20 and these were in turn eliminated using Euler-MacLaurin implemented in MPFI at 100 bits of precision. We can therefore state;

Theorem 9.2. For every Dirichlet L-function of primitive character modulus $q \leq 2,000,000$, we have

$$L_{\chi}(1/2) \neq 0. \quad \square$$

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References

- [1] Tom M. Apostol. *Introduction to Analytic Number Theory*. Undergraduate Texts in Mathematics. Springer, 1976.
- [2] L. Bluestein. A linear filtering approach to the computation of discrete Fourier transform. *IEEE Transactions on Audio and Electroacoustics*, 18(4):451–455, 1970.
- [3] Andrew R. Booker. Artin's conjecture, Turing's method and the Riemann hypothesis. *Experiment. Math.*, 15(4):385–407, 2006.
- [4] William L. Briggs and Van Emden Henson. *The DFT: An Owners Manual for the Discrete Fourier Transform*. SIAM, 1995.
- [5] J.L. Brown Jr. On the Error in Reconstructing a Non-Bandlimited Function by Means of the Bandpass Sampling Theorem. *J. Math. Anal. Appl.*, 18(1):75–84, 1967.
- [6] X. Gourdon. The 10^{13} First Zeros of the Riemann Zeta Function, and Zeros Computation at Very Large Height. <http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf>, 2010.
- [7] H.A. Helfgott. Minor arcs for Goldbach's problem. *arXiv preprint arXiv:1205.5252*, 2012.
- [8] H.A. Helfgott. Major arcs for Goldbach's problem. *arXiv preprint arXiv:1305.2897*, 2013.
- [9] IEEE. IEEE Standard for Binary Floating-Point Arithmetic, IEEE Std 754-1985., 1985.

- [10] B. Lambov. *Reliable Implementation of Real Number Algorithms: Theory and Practice*, chapter Interval Arithmetic Using SSE-2. Lecture Notes in Computer Science. Springer, 2008.
- [11] R.E. Moore. *Error in digital computation*, volume I, pages 61–130. Wiley, 1965.
- [12] R.E. Moore. *Interval analysis*, volume 60. Prentice-Hall Englewood Cliffs, New Jersey, 1966.
- [13] J.M. Muller. *Correctly Rounded Mathematical Library*. <http://lipforge.ens-lyon.fr/www/crlibm/>, 2010.
- [14] H. Rademacher. On the Phragmén-Lindelöf theorem and some applications. *Math. Z.*, 72(1):192–204, 1959.
- [15] N. Revol and F. Rouillier. A library for arbitrary precision interval arithmetic. In *10th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic, and Validated Numerics*, 2002.
- [16] R. Rumely. Numerical Computations Concerning the ERH. *Math. Comp.*, 61(203):415–440, 1993.
- [17] T. Trudgian. Improvements to Turing’s method. *Math. Comp*, 80(276):2259–2279, 2011.
- [18] Alan M. Turing. Some calculations of the Riemann zeta-function. *Proc. Lond. Math. Soc.*, 3(3):99–117, 1953.
- [19] J.S. Walker. *Fast Fourier Transforms*. CRC press Boca Raton, 1991.