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HECKE EIGENVALUES AND RELATIONS FOR DEGREE $n$
SIEGEL EISENSTEIN SERIES OF SQUARE-FREE LEVEL

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Abstract. We describe a basis of Siegel Eisenstein series of degree $n$, square-free level $N$ and arbitrary character $\chi$; then, without using knowledge of their Fourier coefficients, we evaluate the action of the Hecke operators $T(q), T_j(q^2)$ $(1 \leq j \leq n)$ for primes $q|N$. We find the space of Siegel Eisenstein series with square-free level has a basis of simultaneous eigenforms for these operators, and we compute the eigenvalues, thereby obtaining a multiplicity-one result. We then compute the action of the Hecke operators $T(p), T_j(p^2)$ on a basis of Siegel Eisenstein series of level $N \in \mathbb{Z}_+$ provided $4 \nmid N$ and $p$ is a prime with $p \nmid N$, and from this construct a basis of simultaneous eigenforms.

§1. Introduction

Remark that space of Eisenstein series is invariant under Hecke operators

\begin{equation*}
\Gamma_{\infty}^+ \setminus \mathbb{R}^n
\end{equation*}

Refer to notation $E_k^{(n)}(N, \chi)$

§2. Defining Siegel Eisenstein series

For $k, n, N \in \mathbb{Z}_+$ and $\chi$ a character modulo $N$, we want to define a degree $n$, weight $k$, level $N$ Eisenstein series with character $\chi$ for each element of the quotient $\Gamma_{\infty} \backslash \mathbb{R}^n / \Gamma_0(N)$. Given $\gamma \rho \in \mathbb{R}^n$, the natural object to define is

\begin{equation*}
E_{\rho}(\tau) = \sum_{\gamma} \chi(\rho) 1(\gamma) | \gamma \rho |^{-k}
\end{equation*}

where $\gamma \in \Gamma_0(N)$ varies so that $\Gamma_{\infty} \gamma \rho \gamma$ varies over the (distinct) elements of $\Gamma_{\infty} \gamma \rho \Gamma_0(N)$, and

\begin{equation*}
1(\tau) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(C \tau + D)^{-k}
\end{equation*}

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for \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n)(\mathbb{Z}) \). If well-defined, this series converges absolutely uniformly on compact subsets of \( \mathcal{H}(n) \) provided \( k \geq n + 2 \) (and hence is analytic).

[??? it is majorised by the level 1 Eisenstein series in the case \( k \) is even; what about when \( k \) is odd??]

Hence we assume \( k \geq n + 2 \). However, defined as above, \( E_\rho \) may not be well-defined. Thus we over-sum, producing a well-defined function \( E_\rho' \) that is 0 whenever the above sum for \( E_\rho \) is not well-defined, and is a multiple of \( E_\rho \) when \( E_\rho \) is well-defined.

Note that when \( \gamma \in \Gamma_\infty^+ \), \( (1(\tau)|\gamma = 1(\tau)) \). Thus taking \( \gamma^* \in \Gamma(N) \) so that

\[
\Gamma_\infty^+ \Gamma(N) = \bigcup_i \Gamma_\infty^+ \gamma_i^* \text{ (disjoint)},
\]

and setting

\[
E^*(\tau) = \sum_j 1(\tau)|\gamma_j^*;
\]

\( E^* \) is well-defined (and converges absolutely uniformly on compact subsets, so is analytic). With

\[
\Gamma_\rho^+ = \{ \gamma \in \Gamma_0(N) : \Gamma_\infty^+ \Gamma(N) \gamma \rho = \Gamma_\infty^+ \Gamma(N) \gamma \rho \},
\]

take \( \delta_i \in \Gamma_0(N) \), \( \delta'_\ell \in \Gamma_\rho^+ \) so that

\[
\Gamma_0(N) = \bigcup_i \Gamma_\rho^+ \delta_i \text{ (disjoint)}, \quad \Gamma_\rho^+ = \bigcup_\ell \Gamma(N) \delta'_\ell \text{ (disjoint)}
\]

(note that \( \Gamma(N) \subseteq \Gamma_\rho^+ \)). Thus

\[
\Gamma_0(N) = \bigcup_i \Gamma_\rho^+ \delta_i \text{ (disjoint)}.
\]

Set \( G_\pm = \begin{pmatrix} I_{n-1} & -1 \\ 0 & 1 \end{pmatrix} \), \( \gamma_\pm = \begin{pmatrix} G_\pm \\ G_\pm \end{pmatrix} \); remembering \( \Gamma(N) \) is a normal subgroup of \( Sp(n)(\mathbb{Z}) \), we have

\[
\Gamma_\infty \gamma_\rho \Gamma_0(N) = \bigcup_{i,\ell} (\Gamma_\infty^+ \gamma_\rho \Gamma(N) \delta'_\ell \delta_i \cup \Gamma_\infty^+ \gamma_\pm \gamma_\rho \Gamma(N) \delta'_\ell \delta_i) = \bigcup_{i,\ell} (\Gamma_\infty^+ \Gamma(N) \gamma_\rho \delta'_\ell \delta_i \cup \Gamma_\infty^+ \Gamma(N) \gamma_\pm \gamma_\rho \delta'_\ell \delta_i).
\]

Now set

\[
E'_\rho = \sum_{i,\ell} \chi(\delta'_\ell \delta_i) E^*|\gamma_\rho \delta'_\ell \delta_i + \sum_{i,\ell} \chi(\gamma_\pm \delta'_\ell \delta_i) E^*|\gamma_\pm \gamma_\rho \delta'_\ell \delta_i.
\]

Since \( \Gamma_\infty^+ \Gamma(N) \gamma_\pm = \gamma_\pm \Gamma_\infty^+ \Gamma(N) \), we have

\[
E^*|\gamma_\pm = (-1)^k E^*;
\]

hence \( E'_\rho = 0 \) if \( \chi(-1) \neq (-1)^k \).
Assume now that $\chi(-1) = (-1)^k$. Then, since $\Gamma_+^+\Gamma(N)\gamma_0\delta_0 = \Gamma_+^+\Gamma(N)\gamma$, we have $E^*|\gamma \delta_0 = E^*|\gamma$, and hence

$$E'_\rho = 2 \left( \sum_\ell \overline{\chi}(\delta_\ell) \right) \sum_i \overline{\chi}(\delta_i) E^*|\gamma \delta_i.$$ 

Here $\delta_\ell$ varies over a set of representatives for the group $\Gamma(N)\Gamma_\rho^+$ (and we know $\chi$ is trivial on $\Gamma(N)$), so unless $\chi$ is trivial on $\Gamma_\rho^+$, we have $E'_\rho = 0$.

Note that $\gamma \in \Gamma(N)$ if and only if $N \leq 2$. So when $N \leq 2$, we have $\Gamma_\infty \gamma_j$ varying twice over the distinct elements of $\Gamma_\infty \backslash \Gamma_\infty \Gamma(N)$, and

$$E^* = E^*|\gamma \pm = (-1)^k E^*.$$ 

Hence when $N \leq 2$ and $k$ is odd, $E^* = 0$, and thus $E'_\rho = 0$. When $N > 2$ or $k$ is even,

$$\lim_{\tau \to i \infty} E^*(\tau) = \left\{ \begin{array}{ll} 2 & \text{if } N \leq 2, \\ 1 & \text{if } N > 2, \end{array} \right.$$ 

and $\lim_{\tau \to i \infty} E'_\rho(\tau)|\gamma_0^{-1} = 2|\Gamma_0(N) : \Gamma_\rho^+| \lim_{\tau \to i \infty} E^*(\tau)$. (see §4 [Freitag, 1996]).

Also, with $\gamma_j = \gamma_0^{-1} \gamma_0 \gamma_j$, we have

$$\Gamma_\infty \gamma_0 \Gamma_0(N) = \bigcup_{i,j} \Gamma_\infty \gamma_i^{-1} \gamma_j \delta_i = \bigcup_{i,j} \Gamma_\infty \gamma_0 \gamma_j \delta_i.$$

(The above unions over $i, j$ are disjoint when $N > 2$.)

Thus we have proved the following.

**Proposition 2.1.** Assume $\chi(1) = (-1)^k$.

1. For $\gamma_0 \in Sp_n(\mathbb{Z})$, $E_\rho$ is well-defined if and only if $\chi$ is trivial on $\Gamma_\rho^+$. When well-defined, $E_\rho$ is a nonzero multiple of $E'_\rho$, and $E'_\rho \neq 0$ when $N > 2$ or $k$ is even.

2. Suppose $N \leq 2$ and $k$ is odd. Then $E'_\rho = 0$, so either $E_\rho$ is not well-defined or $E_\rho = 0$.

Next we give a description of a convenient choice of representatives corresponding to the Eisenstein series.

**Proposition 2.2.** For any $\gamma \in Sp_n(\mathbb{Z})$, there exists some $\gamma_0 = \begin{pmatrix} I & 0 \\ M_\rho & I \end{pmatrix} \in Sp_n(\mathbb{Z})$ so that $\gamma \in \Gamma_\infty \gamma_0 \Gamma_0(N)$. When $N$ is square-free, take $\rho = (N_0, \ldots, N_n)$ to be a (degree $n$) multiplicative partition of $N$, meaning $N_0 \cdots N_n = N$. Take $M_\rho$ diagonal so that $M_\rho \equiv \begin{pmatrix} I_d & 0 \\ 0 & I \end{pmatrix}$ (for each prime $q$ dividing $N_d$ ($0 \leq d \leq n$)); then as $\rho$ varies, $\gamma_0$ varies over a set of representatives for $\Gamma_\infty \backslash Sp_n(\mathbb{Z})/\Gamma_0(N)$. Further, when $N$ is square-free and $\gamma = \begin{pmatrix} * & * \\ M & N \end{pmatrix} \in Sp_n(\mathbb{Z})$, we have $\gamma \in \Gamma_\infty \gamma_0 \Gamma_0(N)$ if and only if $\text{rank}_q M = \text{rank}_q M_\rho$ for each prime $q | N$ (where $\text{rank}_q M$ denotes the rank of $M$ modulo $q$).
Given $\gamma = \left( \begin{array}{cc} * & * \\ M & N \end{array} \right) \in Sp_n(\mathbb{Z})$, note that we have $\gamma \in \Gamma_\infty \gamma_0 \Gamma_0(\mathcal{N})$ if and only if $(M, I) \in GL_n(\mathbb{Z})(M, N)\Gamma_0(\mathcal{N})$. We proceed algorithmically to first construct a pair $(M', N') \in GL_n(\mathbb{Z})(M, N)\Gamma_0(\mathcal{N})$ with $N' \equiv I (\mathcal{N})$.

Fix a prime $q$ dividing $\mathcal{N}$ with $q^l \parallel \mathcal{N}$. By Lemma ??, we can choose $E_0, G_0 \in SL_n(\mathbb{Z})$ so that $E_0, G_0 \equiv I (\mathcal{N}/q^l)$ and $E_0 N' G_0^{-1} \equiv \left( \begin{array}{cc} N_1 & 0 \\ 0 & 0 \end{array} \right) (q^l)$ where $N_1$ is $d \times d$ and invertible modulo $q$ (so $d = \text{rank}_q \mathcal{N}$). We can adjust $E_0, G_0$ so that $N_1 \equiv \left( \begin{array}{cc} a \\ I \end{array} \right) (q^l)$, some $a$. Similarly, we can choose $\left( \begin{array}{cc} u & v \\ w & x \end{array} \right) \in SL_2(\mathbb{Z})$ so that $\left( \begin{array}{cc} u & v \\ w & x \end{array} \right) \equiv I (\mathcal{N}/q^l)$, $\left( \begin{array}{cc} u & v \\ w & x \end{array} \right) \equiv \left( \begin{array}{cc} a & 0 \\ 0 & \pi \end{array} \right) (q^l)$. Then

$$\gamma_0 = \left( \begin{array}{cc} u & v \\ w & x \end{array} \right) \in \Gamma_0(\mathcal{N})$$
and $E_0 (M, N) \left( G_0 \right. \left. t G_0^{-1} \right) \gamma_0 \equiv \left( \begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array} \right) \left( \begin{array}{cc} I_d & 0 \\ 0 & 0 \end{array} \right) (q^l)$ with $M_1$ $d \times d$.

By symmetry, $M_4 \equiv 0 (q^l)$; since $(M, N) = 1$, $M_3$ is invertible modulo $q$. Thus we can find $E_1, G_1' \in SL_{n-d}(\mathbb{Z})$ so that $E_1, G_1' \equiv I (\mathcal{N}/q^l)$,

$$M_4 = E_1' M_4 G_1' \equiv \left( \begin{array}{cc} I \\ a' \end{array} \right) (q^l).$$

Take $E_1 = \left( \begin{array}{cc} I_d \\ E_1' \end{array} \right)$, $G_1 = \left( \begin{array}{cc} I_d & G_1' \end{array} \right)$, $W = \left( \begin{array}{cc} 0_d & I_{n-d-1} \\ \pi & \pi \end{array} \right)$ where $\pi a' \equiv 1 (q^l)$; then

LYNNE: CHECK THIS

$$(C \ D) = E_1 E_0 (M, N) \left( G_0 \right. \left. t G_0^{-1} \right) \gamma_0 \left( G_1 \right. \left. t G_1^{-1} \right) \left( I \right. \left. W \right)$$

$$\equiv \left( \begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array} \right) I (q^l),$$

and $(C \ D) \in GL_n(\mathbb{Z})(M, N)\Gamma_0(\mathcal{N})$ with $(C \ D) \equiv (M, N) (\mathcal{N}/q^l)$ and $D \equiv I (q^l)$.

Next, suppose $p$ is another prime dividing $\mathcal{N}$ with $p^r \parallel \mathcal{N}$. Applying the above process to the pair $(C \ D)$, we obtain a pair $(C', D') \in GL_n(\mathbb{Z})(M, N)\Gamma_0(\mathcal{N})$ with $(C' \ D') \equiv (M, N) (\mathcal{N}/(q^l p^r))$ and $D' \equiv I (q^l p^r)$.

Continuing, we obtain $(M' \ N') \in$
where $(N M' N')$ is a coprime symmetric pair, so there exist $K', L'$ so that $N|L'$ and $(K' \begin{pmatrix} L' & M' \\ N' & \end{pmatrix}) \in Sp_n(\mathbb{Z})$; note that we must have $K' \equiv I (N)$ since $L' \equiv 0 (N)$ and $N' \equiv I (N)$. Since $M'$ is necessarily symmetric modulo $N$, we can choose a symmetric matrix $M''$ so that $M'' \equiv M' (N)$; set
\[
\delta = \begin{pmatrix} \gamma^t N' & -t L' \\ -t M' & t K' \end{pmatrix} \begin{pmatrix} I & 0 \\ M'' & I \end{pmatrix}.
\]
Then $\delta \in \Gamma(N)$, and $(M'' I) = (M' N') \delta \in GL_n(\mathbb{Z})(M N)\Gamma_0(N)$.

Now suppose $N$ is square-free and $M$ is an integral symmetric matrix. We show that there is some $(M' N') \in GL_n(\mathbb{Z})(M I)\Gamma_0(N)$ so that $N' \equiv I (N)$ and $M' \equiv M_\rho (N)$ where $M_\rho$ is diagonal and, for each prime $q$ dividing $N$, $M_\rho \equiv \begin{pmatrix} I_d \\ 0 \end{pmatrix}$ (q) where $d = \text{rank}_q M$. Then the argument of the preceding paragraph gives us $(M_\rho I) \in GL_n(\mathbb{Z})(M I)\Gamma_0(N)$. So it suffices now to show that for each prime $q|N$, there are $E \in SL_n(\mathbb{Z})$, $\gamma \in \Gamma_0(N)$ so that $E, \gamma \equiv I (N/q)$, and $E(M I) \gamma \equiv (C I) (q)$ where $C = \begin{pmatrix} I_d \\ 0 \end{pmatrix}$ with $d = \text{rank}_q M$.

If $\text{rank}_q M = 0$ then there is nothing to do. Suppose not; first consider the case that $q$ is odd. By §92 of [O'M], we know there exists $E' \in SL_n(\mathbb{Z})_q$ so that $E'M^t E'$ is diagonal with $E'M^t E' \equiv \begin{pmatrix} M_1 \\ 0 \end{pmatrix}$ (q), $M_1 = \begin{pmatrix} a & \\ 0 & 1 \end{pmatrix}$ with $q \nmid a$. Thus we can find $E \in SL_n(\mathbb{Z})$ so that $E \equiv I (N/q)$, $E \equiv E' (q)$. Then
\[
E(M I) \begin{pmatrix} \gamma^t E \\ E^{-1} \end{pmatrix} = (M' I)
\]
where $M' \equiv (E'M^t E') (q)$. Take $\begin{pmatrix} u \\ w \\ v \\ x \end{pmatrix} \in SL_2(\mathbb{Z})$ so that $\begin{pmatrix} u \\ w \\ v \\ x \end{pmatrix} \equiv I (N/q), \begin{pmatrix} u \\ w \\ v \\ x \end{pmatrix} \equiv \begin{pmatrix} \bar{\pi} \\ 0 \\ \bar{\pi} - 1 \\ a \end{pmatrix} (q)$. Set
\[
\gamma = \begin{pmatrix} u & v \\ w & I_{n-1} \\ v & 0 \\ x & 0 \\ 0 & I_{n-1} \end{pmatrix}.
\]
Then $\gamma \equiv I (N/q)$ and $(M' I) \gamma \equiv (C I) (q)$ where $C = \begin{pmatrix} I_d \\ 0 \end{pmatrix}$.

Now suppose $q = 2$. By Lemma ?? there is some $E \in SL_n(\mathbb{Z})$ so that $E \equiv I (N/q)$ and $E M^t E \equiv \begin{pmatrix} M_1 \\ 0 \end{pmatrix}$ (q), where either $M_1 = I_d$ or $M_1 = A_1, A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (d×d where $d = \text{rank}_q M$). In the first case, we are done.
Otherwise, take $A \in SL_n(\mathbb{Z})$ so that $A \equiv I \pmod{N/q}$ and $A \equiv \begin{pmatrix} A_1 & \ldots & A_d \\ I_{n-d} & \end{pmatrix} \pmod{q}$; set 
$\gamma = \begin{pmatrix} tE & E(A-I) \\ E^{-1}A & \end{pmatrix}$. Thus $\gamma \in \Gamma_0(N)$, $\gamma \equiv I \pmod{N/q}$, and $E(M)\gamma \equiv (C I) \pmod{q}$ where $C = \begin{pmatrix} I_d \\ 0 \end{pmatrix}$. \hfill $\Box$

**Proposition 2.3.** Suppose $N$ is square-free, $\chi$ is a character modulo $N$ so that $\chi(-1) = (1)^k$, and $\rho = C_0 \cdots C_n$ is a multiplicative partition of $N$ (as in Proposition 2.2; so $N_0 \cdots N_n = N$). Then $E_\rho$ is well-defined if and only if $\chi^2 = 1$ for all primes $q|N/(N_0N_n)$.

**Proof.** Suppose $q$ is a prime dividing $N_d$ where $0 < d < n$. Fix $\alpha \in \mathbb{F}^\times_q$. By Lemma ??, there exist $G = \begin{pmatrix} u & v \\ w & x \end{pmatrix}, G' = \begin{pmatrix} u' & v' \\ w' & x' \end{pmatrix} \in SL_2(\mathbb{Z})$ so that $G, G' \equiv I \pmod{N/q}$, 
$G \equiv \begin{pmatrix} \pi & \pi - \alpha \\ 0 & \alpha \end{pmatrix} \pmod{q}, G' \equiv \begin{pmatrix} \pi & 0 \\ 0 & \alpha \end{pmatrix} \pmod{q}$.

Let $A, B, C, D, E, W$ be the $n \times n$ matrices 

$A = \begin{pmatrix} u & v \\ w & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & v' \\ w' & 0 \end{pmatrix}, \quad C = \begin{pmatrix} w & 0 \\ 0 & x' \end{pmatrix}$,

$D = \begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}, \quad E = \begin{pmatrix} u' & v' \\ w' & x' \end{pmatrix}, \quad W = \begin{pmatrix} x^2 - 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Then $\gamma' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(N), E \in SL_n(\mathbb{Z})$, and 

$\delta = \begin{pmatrix} E & tE^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \in \Gamma_0^+$. 

Further, $\delta \gamma \gamma' \equiv \gamma \pmod{N}$. Set $\gamma'' = (\delta \gamma \gamma')^{-1} \gamma$. So $\gamma'' \in \Gamma(N), \quad \gamma'' \equiv \gamma \hfill \Gamma(N)$, with $\chi(\gamma'' \gamma) = \chi^2(\alpha)$. Thus the condition that $\chi^2 = 1$ for all primes $q|N/(N_0N_n)$ is necessary for $E_\rho$ to be well-defined.

Now suppose $\chi^2 = 1$ for all primes $q|N/(N_0N_n)$, and suppose $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^+$. Thus there exist $\delta = \begin{pmatrix} tE & W \\ E & I \end{pmatrix} \in \Gamma_0^+, \gamma' \in \Gamma(N)$ so that $\delta \gamma \gamma' \gamma = \gamma$. Fix a prime $q|N_d, 0 \leq d \leq n$.

When $d = 0$, we have $ED \equiv I \pmod{q}$, so $\det D \equiv \det E \equiv 1 \pmod{q}$ and $\chi_q(\det D) = 1$. When $d = n$, we have $EA \equiv I \equiv A^t D \pmod{q}$, so $\det D \equiv \det E \equiv 1 \pmod{q}$ and $\chi_q(\det D) = 1$. 


Now suppose $0 < d < n$. Write

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$$

where $A_1, D_1, E_1$ are $d \times d$. Then we have $E_3(A_1 A_2) \equiv 0 \ (q)$; since the rows of $(A_1 A_2)$ are linearly independent modulo $q$, we must have $E_3 \equiv 0 \ (q)$. Also,

$$E_1(A_1 A_2) \equiv (I_d \ 0) \ (q), \quad E_4(D_3 D_4) \equiv (0 \ I_n-d) \ (q),$$

so $A_2, D_3 \equiv 0 \ (q), \ A_1 \equiv \overline{E}_1 \ (q), \ D_4 \equiv \overline{E}_4 \ (q)$. Since $A^t D \equiv I \ (q)$, we must have $D_1 \equiv t E_1 \ (q)$. Thus we have

$$\det D \equiv \det E_1 \cdot \det \overline{E}_4 \equiv (\det E_1)^2 \ (q)$$

and

$$\chi_q(\det D) = \chi_2^2(\det E_1) = 1.$$

Consequently $\chi(\gamma) = \chi(\det D) = 1$, and hence the condition that $\chi_2 = 1$ for all primes $q|N/(N_0 N_n)$ is sufficient for $E_\rho$ to be well-defined. □

We now give a robust definition of $E_\rho$.

**Definition.** Having fixed $n, k, N \in \mathbb{Z}_+$ with $k \geq n + 2$, $\chi$ a character modulo $N$, and $\gamma_\rho \in \text{Sp}_n(\mathbb{Z})$, we define

$$E_\rho = \begin{cases} \frac{1}{2^{|\Gamma_0(N) : \Gamma_\infty \Gamma_\rho^+\Gamma_0(N)|}} E'_{\rho} & \text{if } N > 2, \\ \frac{1}{4^{|\Gamma_0(N) : \Gamma_\infty \Gamma_\rho^+|}} E'_{\rho} & \text{if } N \leq 2. \end{cases}$$

**Remark.** Suppose that $G_\pm M_\rho = M_\rho G_\pm$. Then for $G \in GL_n(\mathbb{Z})$, $\gamma \in \Gamma_0(N)$, we have $G(M_\rho I) \gamma = G G_\pm (M_\rho I) \gamma_\pm \gamma$. So with $\gamma_\rho = \begin{pmatrix} I & 0 \\ M_\rho & I \end{pmatrix}$, we have $\Gamma_\infty \Gamma(N) \gamma_\rho \gamma = \Gamma_\infty \Gamma(N) \gamma_\rho \gamma_\pm \gamma$ if and only if $N \leq 2$ (since $\gamma_\pm \in \Gamma(N)$ if and only if $N \leq 2$). Thus,

$$E_\rho(\tau) = m_\rho \sum_\gamma \overline{\chi}(\gamma) 1(\tau)|\gamma_\rho \gamma$$

where $\gamma$ varies so that $\Gamma_\infty^+ \gamma_\rho \Gamma_0(N) = \cup_\gamma \Gamma_\infty^+ \gamma_\rho \gamma$ (disjoint), and

$$m_\rho = \begin{cases} 1 & \text{if } N \leq 2, \\ \frac{1}{2} & \text{otherwise}. \end{cases}$$

**Lynne:** This next defined earlier?

We let $E_\rho^{(n)}(N, \chi)$ denote the space spanned by these forms.
§3. Defining Hecke operators

For each prime $p$, we define Hecke operators $T(p)$, $T_j(p^2)$ ($1 \leq j \leq n$) acting on Siegel modular forms; then we describe explicit sets of matrices that give the action of these operators.

Fix a prime $p$; set $\Gamma = \Gamma_0(N)$ and take $f \in M_k^{(n)}(N, \chi)$. We define

$$f | T(p) = p^{(k-n-1)/2} \sum_\gamma \chi(\gamma) f|_{\delta^{-1} \gamma}$$

where $\delta = \left( \begin{array}{cc} pI_n \\ I_n \end{array} \right)$, $\gamma$ varies over $(\delta \Gamma \delta^{-1} \cap \Gamma) \Gamma$, and for $\gamma' = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in Sp_n(\mathbb{Z})$,

$$f(\tau)|\gamma' = (\det \gamma')^{k/2} \det(C\tau + D)^{-k} f((A\tau + B)(C\tau + D)^{-1}).$$

We define

$$f | T_j(p^2) = p^{j(k-n-1)} \sum_\gamma \chi(\gamma) f|_{\delta_j^{-1} \gamma}$$

where $\delta_j = \left( \begin{array}{cc} X_j & 0 \\ 0 & X_j^{-1} \end{array} \right)$, $X_j = \left( \begin{array}{cc} pI_j \\ I_{n-j} \end{array} \right)$, and $\gamma$ varies over $(\delta_j \Gamma \delta_j^{-1} \cap \Gamma) \Gamma$.

**Proposition 3.1.** Let $p$ be a prime, $f \in M_k^{(n)}(N, \chi)$. For $0 \leq r, n_0 + n_2 \leq n$, let

$$D_r = \left( \begin{array}{cc} pI_r \\ I \end{array} \right), \quad D_{n_0,n_2} = \left( \begin{array}{cc} pI_{n_0} & I \\ \frac{1}{p}I_{n_2} \end{array} \right) (n \times n),$$

and let

$$K_r = D_r SL_n(\mathbb{Z})D_r^{-1} \cap SL_n(\mathbb{Z}),$$

$$K_{n_0,n_2} = D_{n_0,n_2} SL_n(\mathbb{Z})D_{n_0,n_2}^{-1} \cap SL_n(\mathbb{Z}).$$

Then

$$f | T(p) = p^{(k-n-1)/2} \sum_{0 \leq r \leq n} \chi(p^{n-r}) \sum_{G,Y} f| \left( D_r^{-1} \frac{1}{p}D_r \right) \left( G^{-1} \quad Y^tG \right)$$

where $G$ varies over $SL_n(\mathbb{Z})/K_r$ and $Y$ varies over

$$\mathcal{Y}_r = \left\{ \left( \begin{array}{cc} Y_0 \\ 0 \end{array} \right) \in \mathbb{Z}_{sym}^{n \times n} : Y_0 r \times r, \text{ varying modulo } p \right\}.$$

Also,

$$f | T_j(p^2) = p^{j(k-n-1)} \sum_{n_0 + n_2 \leq j} \chi(p^{j-n_0+n_2}) \sum_{G,Y} f| \left( D_{n_0,n_2}^{-1} \right) \left( G^{-1} \quad Y^tG \right)$$
where \( G \) varies over \( SL_n(\mathbb{Z})/\mathcal{K}_{n_0,n_2} \) and \( Y \) varies over \( \mathcal{Y}_{n_0,n_2} \), the set of all integral, symmetric \( n \times n \) matrices

\[
\begin{pmatrix}
Y_0 & Y_2 & Y_3 & 0 \\
Y_2 & Y_1/p & 0 & 0 \\
Y_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

with \( Y_0 n_0 \times n_0 \), varying modulo \( p^2 \), \( Y_1 (j-n_0-n_2) \times (j-n_0-n_2) \), varying modulo \( p \) provided \( p \nmid \det Y_1 \), \( Y_2 n_0 \times (j-n_0-n_2) \), varying modulo \( p \) and \( Y_3 n_0 \times (n-j) \), varying modulo \( p \).

**Proof.** Fix \( \Lambda = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n \) (a reference lattice).

By Lemma ??, as \( G \) varies over \( SL_n(\mathbb{Z})/\mathcal{K}_r \), \( \Omega = AGD_r \) varies over all lattices \( \Omega \), \( p\Lambda \subseteq \Omega \subseteq \Lambda \) with \([\Lambda : \Omega] = p^r \). Thus by Proposition 3.1 and (the proof of) Theorem 6.1 in [HW], claim (1) of the proposition follows.

For \( \Omega \) another lattice on \( \mathbb{Q} \), let \( \text{mult}_{\{\Lambda, \Omega\}}(a) \) be the multiplicity of the value of \( a \) among the invariant factors \( \{\Lambda : \Omega\} \). By Lemma ??, as \( G \) varies over \( SL_n(\mathbb{Z})/\mathcal{K}_{n_0,n_2} \) \( \Omega = AGD_{n_0,n_2} \) varies over all lattices \( \Omega \), \( p\Lambda \subseteq \Omega \subseteq \frac{1}{p^2}\Lambda \), with \( \text{mult}_{\{\Lambda, \Omega\}}(1/p) = n_2 \), \( \text{mult}_{\{\Lambda, \Omega\}}(p) = n_0 \). Thus by Proposition 3.1 and (the proofs of) Theorems 4.1 and 6.1 in [HW], claim (2) of the proposition follows. □

**Remark.** For \( N' \in \mathbb{Z}_+ \), so that \( p \nmid N' \), we can choose \( G, Y \) in the above proposition so that \( G \equiv I \mod{(N')} \) and \( Y \equiv 0 \mod{(N')} \). Also, if \( p \nmid N \), then

\[
f[M](p) = p^{n(k-n-1)/2} \sum_Y \left( \frac{Y}{p} \right) \left( \frac{I}{p} \right)^{n/2} Y
\]

where \( Y \) varies over \( \mathcal{Y}_n \), and

\[
f[M_{\gamma^t}](p^2) = p^{j(k-n-1)} \sum_{G,Y} \left( \frac{G^{-1}}{p} \right) \left( \frac{Y^tG}{p} \right)
\]

where \( G \) varies over \( SL_n(\mathbb{Z})/\mathcal{K}_{j,0} \) and \( Y \) varies over \( \mathcal{Y}_{j,0} \).

LYNNE: CHECK THESE ABOVE SUMS

§4. Hecke operators on Siegel Eisenstein series of square-free level

Throughout this section, we assume \( N \) is square-free, \( \chi \) is a character modulo \( N \) so that \( \chi(-1) = (-1)^k \); further, we assume either \( N > 2 \) or \( k \) is even. Take a multiplicative partition \( \rho = (N_0, \ldots, N_n) \) of \( N \) (so \( N_0 \cdots N_n = N \) ), and assume that \( \mathbb{Z}_p \neq 0 \) (so by Proposition 2.3, \( \chi_{q^r} \equiv 1 \) for all primes \( q \nmid (N_0N_n) \)). Take diagonal \( M_\rho \) as in Proposition 2.2, \( \gamma_\rho = \begin{pmatrix} I & 0 \\ M_\rho & I \end{pmatrix} \).

With \( \beta = \begin{pmatrix} * & * \\ M & N \end{pmatrix} \in SL_n(\mathbb{Z}) \) and \( \gamma \in \Gamma_0(N) \) so that \( \Gamma_\infty^+ \beta = \Gamma_\infty^+ \gamma_\rho \gamma \), we can determine how to compute \( \chi(\gamma) \) from \( (M, N) \).
Suppose \( \left( \begin{array}{ccc} * & * \\ M & N \end{array} \right) \in \Gamma_\infty \gamma \Gamma_0(N) \); so \((M \ N) = E'(M, I) \gamma \) for some \( E' \in SL_n(\mathbb{Z}) \) and \( \gamma = \left( \begin{array}{cc} A & B \\ \text{C} & D \end{array} \right) \in \Gamma_0(N) \). Fix \( q \) and take \( d = \text{rank}_q M \). Thus \( \text{rank}_q M = d \), so we can find \( E, G \in SL_n(\mathbb{Z}) \) so that \( EGM \equiv \left( \begin{array}{cc} M_1 & 0 \\ 0 & 0 \end{array} \right) \ Qt \). Thus \( \text{rank}_q M = d \), so we can find \( E, G \in SL_n(\mathbb{Z}) \) so that \( EGM \equiv \left( \begin{array}{cc} M_1 & 0 \\ 0 & 0 \end{array} \right) \ Qt \). Hence

\[
EMG \equiv \left( \begin{array}{cc} M_1 & 0 \\ 0 & 0 \end{array} \right) \equiv EE' \left( \begin{array}{cc} I_d & 0 \\ 0 & 0 \end{array} \right) AG \ (q),
\]

\[
\left( \begin{array}{cc} N_1 & N_2 \\ 0 & N_4 \end{array} \right) \equiv EE' \left( \begin{array}{cc} I_d & 0 \\ 0 & 0 \end{array} \right) B + D \ (q). \]

Given the shape of \( EMG \), we must have \( EE' \equiv \left( \begin{array}{cc} E_1 & E_2 \\ E_3 & E_4 \end{array} \right) \ Qt \) where \( E_1 \) is \( d \times d \) and \( E_1, E_4 \) are invertible modulo \( q \), and then \( AG \equiv \left( \begin{array}{cc} A_1 & 0 \\ A_3 & A_4 \end{array} \right) \ Qt \) where \( A_1 \) is \( d \times d \); since \( N|C \), \( A_1, A_4 \) are invertible modulo \( q \). We have \( A\ D \equiv I \ Qt \), so

\[
D \ (q). \]

Further, we must have

\[
A_1 \cdot \text{det} \ M_1 \cdot \text{det} \ N_4 \equiv \text{det} \ E_1 \cdot \text{det} \ E_4 \cdot \text{det} \ A_1 \cdot \text{det} \ D_4 \equiv (\text{det} \ E_1)^2 \cdot \text{det} D \ (q). \]

Note that when \( d = 0 D \equiv N \ Qt \), and when \( d = n, \ A \equiv A \ Qt \). When \( 0 < d < n \), we have \( \chi_q = 1 \) so

\[
\chi_q(\text{det} \ M_1 \cdot \text{det} \ N_4) = \chi_q(\text{det} D). \]

Thus we can define \( \chi_q(M, N) = \chi_q(\text{det} \ M_1 \cdot \text{det} \ N_4) \), and

\[
\chi(M, N) = \prod_{q \mid N} \chi_q(M, N). \]

Then we have

\[
E_\rho(\tau) = \frac{1}{2} \sum_{(M \ N)} \chi(M, N) \det(M \tau + N)^{-k} \]

where \( (M \ N) \) varies over coprime symmetric pairs so that

\[
SL_n(\mathbb{Z})(M, I) \Gamma_0(N) = \cup (M \ N) SL_n(\mathbb{Z})(M \ N) \ (\text{disjoint}). \]

Now we prove the following.
Theorem 4.1. Fix a prime $q|\mathcal{N}$, and fix a multiplicative partition $\sigma = (N_0', \ldots, N_n')$ of $N/q$. For $0 \leq d \leq n$, let $E_{\sigma,d}$ denote $E_{\rho'}$ where $\rho' = (N_0, \ldots, N_n)$.

Then

$$E_{\sigma,d}|T(q) = q^{kd - (d+1)/2} \chi_{N/q} \left( \begin{pmatrix} I_d & \frac{1}{q} I_{n-d} \\ \frac{1}{q} I_{n-d} & I_{n-d} \end{pmatrix} \right) M_{\sigma,d} \left( \begin{pmatrix} qI_d & I_{n-d} \\ I_{n-d} & I_{n-d} \end{pmatrix} \right)$$

$$\cdot \sum_{t=0}^{n-d} q^{-dt - (t-1)/2} \beta(d + t, t) \text{sym}_t(t) \prod_{\sigma_{d+t}}^{}$$

where

$$\text{sym}_t(t) = \sum_U \chi_q(detU),$$

$U$ varying over $\mathbb{F}_{st}$.

Remark. In Lemma ?? we evaluate $\text{sym}_t(t)$.

?? WHAT IF $n - \ell = 0$ and $\chi_1 \neq 0$? Have $E_0 = 0$ for $0 < t < n$. How do we modify this argument to get $E_0|T(q) = E_0 + * E_n$?

Proof.

LYNNE: ?? $n - \ell \mapsto d$??

Write $E_d$ for $E_{\sigma,d}$. We know $E_d(\tau)$ is a sum over representatives for $SL_n(\mathbb{Z})$-equivalence classes of coprime pairs $(M, N)$ with $\text{rank}_q M = d$; we can assume $q$ divides the lower $n - d$ rows of $M$. By Proposition 3.1,

$$E_d(\tau)|T(q) = q^{-n(n+1)/2} \sum_{M,N,Y} \det(M\tau + MY/q + N)^{-k}$$

$$= q^{kn - n(n+1)/2} \sum_{M,N,Y} \det(M\tau + MY + qN)^{-k}$$

where $Y$ varies over $\mathcal{Y}_n$. We have

$$\det(M\tau + MY + qN)^{-k} = q^{-(n-d)} \det(M'\tau + N')^{-k}$$

where

$$(M', N') = \left( \begin{pmatrix} I_d & \frac{1}{q} I_{n-d} \\ \frac{1}{q} I_{n-d} & I_{n-d} \end{pmatrix} (M \text{ } MY + qN).$$

We know the upper $d$ rows of $M$ are linearly independent modulo $q$, as are the lower $n - d$ rows of $N$. Thus $(M', N') = 1$, and $\text{rank}_q M' \geq d$. Also note that

$$\det(M\tau + MY + qN)^{-k} = q^{-(n-d)k} \det(M'\tau + N')^{-k}.$$
Recall that we can assume $Y \equiv 0 \ (N/q)$. Also, we know $E_d$ is supported on the $\Gamma_0(N)$-orbit of $GL_n(\mathbb{Z})(M, I)$. Take $(M, N) = (M, I) \gamma$ where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(N)$. Take a prime $q' \mid N$ and let $d' = \text{rank}_{q'} M$. Choose $E \in SL_n(\mathbb{Z})$ so that $AE \equiv \begin{pmatrix} A_1 & 0 \\ 0 & \ast \end{pmatrix} (q')$ where $A_1$ is $d' \times d'$ (possible since we necessarily have rank$_{q'} A = n$ since $q' \mid N$). Then since $A^t D \equiv I (q')$, we have $D^t E - 1 \equiv \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix} (q')$ with $D_1 d' \times d'$.

Thus $(M, N) \begin{pmatrix} E & t \end{pmatrix}^{-1} \equiv \begin{pmatrix} A_1 & 0 & \ast & \ast \\ 0 & 0 & 0 & D_4 \end{pmatrix} (q')$, and

$(M', N') \begin{pmatrix} E & t \end{pmatrix}^{-1} \equiv \begin{pmatrix} A_1' & 0 & \ast & \ast \\ 0 & 0 & 0 & D_4' \end{pmatrix} (q')$

where, modulo $q'$,

$A_1' \equiv \begin{cases} A_1 & \text{if } d' \leq d, \\ \left( \frac{q}{d'} \right) A_1 & \text{if } d' \geq d; \end{cases}$

$D_4' \equiv \begin{cases} \left( \frac{q}{d'} \right) I_{n-d} & \text{if } d' \leq d, \\ D_4 & \text{if } d' \geq d. \end{cases}$

Therefore

$\chi_{q'}(M', N') = \chi_{q'}(M'E, N't^{-1}E^{-1}) = \chi_{q'}(\det A_1', \det D_4')$

$= \chi_{q'}(d' - d') \cdot \chi_{q'}(\det A_1 \cdot \det D_4),$

$\chi_{q'}(\det A_1 \cdot \det D_4) = \chi_{q'}(M, N),$  

$\chi_{q'}(q^{d' - d'}) = \chi_{q'} \left( \begin{pmatrix} I_d & \frac{1}{q} I_{n-d} \\ \frac{1}{q} I_{n-d} & I_{n-d} \end{pmatrix} M, \begin{pmatrix} qI & 0 \\ 0 & I_{n-d} \end{pmatrix} N \right).$

Hence

$\chi_{q'}(M', N') = \chi_{q'}(M'E, N't^{-1}E^{-1})$

$= \chi_{q'}(\det A_1' \cdot \det D_4')$

$= \chi_{q'} \left( \begin{pmatrix} I & \frac{1}{q} I_{n-d} \\ \frac{1}{q} I_{n-d} & I_{n-d} \end{pmatrix} M, \begin{pmatrix} qI & 0 \\ 0 & I_{n-d} \end{pmatrix} \right) \chi_{q'}(M, N).$

Therefore $\chi_{N'/q}(M, N) = \chi_{N'/q} \left( \begin{pmatrix} I & \frac{1}{q} I_{n-d} \\ \frac{1}{q} I_{n-d} & I_{n-d} \end{pmatrix} M, \begin{pmatrix} qI & 0 \\ 0 & I_{n-d} \end{pmatrix} \right) \chi_{N'/q}(M', N').$
Reversing, take \((M', N')\) a coprime symmetric pair with \(\text{rank}_q M' = d + t\); assume \(E_{\sigma, d + t} \neq 0\). We need to count the equivalence classes \(SL_n(\mathbb{Z})(M, N)\) so that

\[
\begin{pmatrix} \text{Id} & qI_{n-d} \\ \frac{1}{q}I_{n-d} \end{pmatrix} \begin{pmatrix} M & Y \\ qN & q \end{pmatrix} \in SL_n(\mathbb{Z})(M', N').
\]

For any \(E \in SL_n(\mathbb{Z})\), we have \(\begin{pmatrix} \text{Id} & qI_{n-d} \\ qI_{n-d} \end{pmatrix} E \begin{pmatrix} \frac{1}{q}I_{n-d} \\ \text{Id} \end{pmatrix} \in SL_n(\mathbb{Z})\) if and only if \(E \in K_d\). Thus we need to count the number of \(E \in K_d \setminus SL_n(\mathbb{Z})\) and \(Y \in \mathbb{Z}^{n,n}_{\text{sym}}\) (varying modulo \(q\)) so that

\[
(M, N) = \begin{pmatrix} \text{Id} & qI_{n-d} \\ qI_{n-d} \end{pmatrix} E(M' (N' - M'Y)/q)
\]

is a coprime pair. We can assume the top \(d + t\) rows of \(M'\) are linearly independent modulo \(q\), and that \(q\) divides the lower \(n - d - t\) rows of \(M'\). To have \(\text{rank}_q M = d\), we need to choose \(E\) so that the top \(d\) rows of \(EM'\) are linearly independent modulo \(q\); using Lemma ?? there are

\[q^{d(n-d-t)} \beta(d + t, d) = q^{d(n-d-t)} \beta(d + t, t)\]

choices for \(E\). We need to choose \(Y\) so that \(N\) is integral and \((M, N) = 1\); equivalently, for any \(G \in SL_n(\mathbb{Z})\), we need \(N'G^{-1}\) integral and \((MG, N'G^{-1}) = 1\). Using left multiplication by \(K_d\), we can adjust the choice of \(E\) so that the lower \(n - d - t\) rows of \(EM'\) are divisible by \(q\), and then we can choose \(G \in SL_n(\mathbb{Z})\) so that

\[EM'G \equiv \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_5 & 0 \\ 0 & 0 & 0 \end{pmatrix} (q)\]

where \(M_1\) is \(d \times d\), \(M_5\) is \(t \times t\), and \(M_1, M_5\) are invertible modulo \(q\). Write

\[EN'^t G^{-1} = \begin{pmatrix} N_1 & N_2 & N_3 \\ N_4 & N_5 & N_6 \\ N_7 & N_8 & N_9 \end{pmatrix}, G^{-1} Y' G^{-1} = \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 \\ Y_7 & Y_8 & Y_9 \end{pmatrix}\]

where \(N_1, Y_1\) are \(d \times d\) and \(N_5, Y_4\) are \(t \times t\). By symmetry, \(N_7, N_8 \equiv 0 (q)\), and then since \((M', N') = 1\), we must have \(\text{rank}_q N_9 = n - d - t\). Also, as \(Y\) varies over \(F_{\text{sym}}^{n,n}\), so does \(G^{-1} Y' G^{-1}\). To have \(N\) integral, we need \((Y_1 Y_2 Y_3) \equiv M_1 (N_1 N_2 N_3) (q)\). Then by symmetry, we find \(N_4 \equiv M_5 Y_2 (q)\). So to have \((M, N) = 1\), we need \(\text{rank}_q (N_5 - M_5 Y_4) = t\), or equivalently,

\[\text{rank}_q (N_5 - M_5 Y_4)^t M_5 = t.\]

As \(Y_4\) varies over \(F_{\text{sym}}^{t,t}\), so does \(N_5 - M_5 Y_4^t M_5\). We have

\[\chi_q(M, N) = \chi_q(\det M_1 \cdot \det (N_5 - Y_4 M_5) \cdot \det N_9) = \chi_q(\det M_1 \cdot \det M_5 \cdot \det N_9) \cdot \chi_q(\det (N_5 - M_5 Y_4)^t M_5) = \chi_q(M', N') \cdot \chi_q(\det (N_5 - M_5 Y_4)^t M_5).\]
We have no constraints on $Y_5$ and $Y_6$, so as we vary $Y$ subject to the above conditions, we get

\[ \sum_Y \chi_q(M,N) = \chi_q(M',N') \cdot q^{(n-d-t)(n-d+t+1)/2} \sum_{U \in \mathcal{F}_{\text{sym}}} \chi_q(\det U) \]

\[ = \chi_q(M',N') q^{(n-d-t)(n-d+t+1)/2} \text{sym}_q^3(t), \]

as claimed. □

This theorem allows us to diagonalise the space of Eisenstein series. To aid in our description of this, we define a partial ordering on multiplicative partitions of $N$, as follows.

**Definition.** For $\rho, \beta$ multiplicative partitions of $N$ and $\mathcal{Q}|N$, we write $\beta = \rho(\mathcal{Q})$ if, for every prime $q|\mathcal{Q}$, we have $\text{rank}_q M_{\beta} = \text{rank}_q M_{\rho}$. Similarly, we write $\beta > \rho(\mathcal{Q})$ if, for every prime $q|\mathcal{Q}$, we have $\text{rank}_q M_{\beta} > \text{rank}_q M_{\rho}$.

**Corollary 4.2.** Let $q$ be a prime dividing $N$. For $\rho$ a partition of $N$ so that $E_\rho \neq 0$, there are $a_{\rho,\alpha}(q) \in \mathbb{C}$ so that $a_{\rho,\rho}(q) = 1$ and

\[ \sum_{\alpha \geq \rho(\mathcal{Q})} a_{\rho,\alpha}(q) E_\alpha \]

is an eigenform for $T(q)$ with eigenvalue

\[ \lambda_\rho(q) = q^{kd-d(d+1)/2} \chi_{N/q} \left( \begin{pmatrix} I_d \\ \frac{1}{q} I \end{pmatrix} M_\rho, \begin{pmatrix} qI_d \\ I \end{pmatrix} \right) \]

where $d = \text{rank}_q M_\rho$. Further, suppose $\alpha = \rho(N/q)$, $\alpha > \rho(q)$, with $d = \text{rank}_q M_\rho$, $d + t = \text{rank}_q M_\alpha$; then we have $a_{\rho,\alpha}(q) \neq 0$ if and only if either (1) $\chi_q = 1$, or (2) $\chi_q^2 = 1$ and $t$ is even.

**Proof.** By Lemma ?? sym$_q^3(t) = 0$ if and only if (1) $\chi_q^2 \neq 1$, or (2) $\chi_q \neq 1$ and $t$ is odd. Thus by Theorem 4.1,

\[ \text{span}\{ E_\alpha : \alpha = \rho(N/q), \alpha \geq \rho(q), \text{ so that either (1) } \chi_q = 1, \text{ or (2) } \chi_q^2 = 1 \text{ and rank}_q M_\alpha - \text{rank}_q M_\rho \text{ is even } \} \]

is invariant under $T(q)$, and the matrix for $T(q)$ on this basis is upper triangular with nonzero upper triangular entries. Then the standard process of diagonalising an upper triangular matrix yields the result. □

We now prove a multiplicity-one result for the Eisenstein series of square-free level.
Corollary 4.3. Suppose $E_{\rho} \neq 0$. For $\alpha \geq \rho \ (Q)$ and prime $q\mid Q$, set $a_{\rho, \alpha}(q) = a_{\rho, \sigma}(q)$ where $\sigma = \rho \ (N/q)$, $\sigma = \alpha \ (q)$, and set

$$a_{\rho, \alpha}(Q) = \prod_{q\mid Q} a_{\rho, \alpha}(q).$$

Then with

$$\tilde{E}_{\rho} = \sum_{\alpha \geq \rho \ (N)} a_{\rho, \alpha}(N) E_{\alpha},$$

for every prime $q\mid N$ we have

$$\tilde{E}_{\rho} T(q) = \lambda_{\rho}(q) \tilde{E}_{\rho}$$

(where $\lambda_{\rho}(q)$ is defined in Corollary 4.2).

Proof. Fix a prime $q\mid N$. For $\alpha \geq \rho \ (N)$, take $\beta = \alpha \ (N/q)$, $\beta = \rho \ (q)$. Then $a_{\rho, \alpha}(N) = a_{\rho, \beta}(N/q) a_{\rho, \alpha}(q)$. Hence

$$\tilde{E}_{\rho} = \sum_{\beta \geq \rho \ (N/q)} a_{\rho, \beta}(N/q) \sum_{\alpha \geq \rho \ (N/q)} a_{\rho, \alpha}(q) E_{\alpha}.$$

We argue that when $a_{\rho, \beta}(N/q) \neq 0$, we have $a_{\rho, \alpha}(q) = a_{\beta, \alpha}(q)$ and $\lambda_{\rho}(q) = \lambda_{\beta}(q)$.

Fix $\beta$ so that $\beta \geq \rho \ (N/q)$, $\beta = \rho \ (q)$, and suppose $a_{\rho, \beta}(N/q) \neq 0$. Take $Q|N/q$ so that $\beta = \rho \ (N/Q)$, $\beta > \rho \ (Q)$. Thus $a_{\rho, \beta}(N/q) = a_{\rho, \beta}(Q)$. Since $a_{\rho, \beta}(Q) \neq 0$, for each prime $q'|Q$ we have either (1) $\chi_{q'} = 1$, or (2) $\chi_{q'}^2 = 1$ and rank$_{q'} M_{\beta} = \text{rank}_{q'} M_{\rho}$ is even.

Suppose $q'$ is a prime dividing $Q$ so that $\chi_{q'} \neq 1$. Set $r = \text{rank}_{q'} M_{\rho}$, $r + t = \text{rank}_{q'} M_{\beta}$ (so $t$ is even). Then for $0 \leq d \leq n$,

$$\chi_{q'} \left( \left( \begin{array}{cc} I_d & \frac{1}{q} I \\ \frac{1}{q} I & I \end{array} \right) M_{\rho}, \left( \begin{array}{c} q I_d \\ I \end{array} \right) \right) = \chi_{q'} \left( \left( \begin{array}{cc} I_d & \frac{1}{q} I \\ \frac{1}{q} I & I \end{array} \right) \right) \left( \begin{array}{c} I_r \\ 0 \end{array} \right), \left( \begin{array}{c} q I_d \\ I \end{array} \right) \right)$$

$$= \begin{cases} \chi_{q'}(q^{r-d}) & \text{if } d \leq r, \\
\chi_{q'}(q^{d-r}) & \text{if } d \geq r \\
\chi_{q'}(q^{d-r}) & \text{if } d = r \end{cases}$$

(since $\chi_{q'}^2$). Similarly,

$$\chi_{q'} \left( \left( \begin{array}{cc} I_d & \frac{1}{q} I \\ \frac{1}{q} I & I \end{array} \right) M_{\beta}, \left( \begin{array}{c} q I_d \\ I \end{array} \right) \right) = \chi_{q'}(q^{d-r-t})$$

and $\chi_{q'}(q^{d-r-t}) = \chi_{q'}(q^{d-r})$ since $t$ is even and $\chi_{q'}^2 = 1$.

For each prime $q'|N/Q$, we either have $\beta = \rho \ (q')$ or $\chi_{q'} = 1$. Thus for $0 \leq d \leq n$,

$$\chi_{N/q} \left( \left( \begin{array}{cc} I_d & \frac{1}{q} I \\ \frac{1}{q} I & I \end{array} \right) M_{\rho}, \left( \begin{array}{c} q I_d \\ I \end{array} \right) \right) = \chi_{N/q} \left( \left( \begin{array}{cc} I_d & \frac{1}{q} I \\ \frac{1}{q} I & I \end{array} \right) M_{\beta}, \left( \begin{array}{c} q I_d \\ I \end{array} \right) \right).$$
Hence $\lambda_\beta(q) = \lambda_\mu(q)$. Further, with $\sigma_d$, $\alpha_d$ partitions of $N$ so that $\sigma_d = \rho(N/q)$, rank$_qM_{\sigma_d} = d$, $\alpha_d = \beta(N/q)$, rank$_qM_{\alpha_d} = d$, the matrix for $T(q)$ on $^t(E_{\sigma_0}, \ldots, E_{\sigma_n})$ is equal to the matrix for $T(q)$ on $^t(E_{\alpha_0}, \ldots, E_{\alpha_n})$, and hence $a_{\rho,\sigma_d}(q) = a_{\beta,\alpha_d}(q)$, $0 \leq d \leq n$. $\square$

Now we evaluate the action of $T_j(q^2)$ on $E_{\rho}$. Note that since the Hecke operators commute, the multiplicity-one result of Corollary 4.3 tells us that each $\tilde{E}$ of $\lambda$ value $a$ Assume Theorem 4.4.

For $0 \leq j, d \leq n$, 
\[
E_{\sigma_d}|T_j(q^2) = \sum_{t=0}^{n-d} A_j(d,t)E_{\sigma_{d+t}};
\]
when $\chi_q = 1$, 
\[
A_j(d,t) = q^{(j-t)d-t(t+1)/2}\beta(d+t,t) \cdot \sum_{d_1=0}^{j-d} \sum_{d_2=0}^{d} q^{b_j(d_1,d_2,d_5)} \chi_{N/q}(D_{d_1,t}M_{\sigma_d}D_j^{-1},D_{d_2,t},D_j) \cdot \beta(d_1,d_2) \beta(t,d_5)\beta(n-d-t, d_1 + n - d - j - d_6) \cdot \beta(t-d_5,d_6)\text{sym}_q^\lambda(t-d_5-d_6)\text{sym}_q^\lambda(d_5,d_6),
\]
where $r = j - d_1 - d_5 + d + 8$, and 
\[
a_j(d,d_1,d_5,d_8) = (k-d)(2d_1 + d_5 - d_8) + d_1(d_1 - d_8 - j - 1) - d_8(d_5 + t) - d_5(d_5 + 1)/2 + d_8(d_8 + 1)/2
\]

[LYNNE: DEFINE sym$^\lambda(b,c)$]
(Note that sym$^\lambda(t-d_5-d_6)$, sym$^\lambda(d_5,d_8)$ are evaluated in Lemmas ???)

Proof. Fix $d = \text{rank}_qM_{\rho}$; to ease some notation later, set $\ell = n - d$. 
\[
E_{n-\ell}|T_j(q^2) = q^{(k-n-1)} \sum_{G,Y} E_{n-\ell} \left( D_j^{-1} \right) \left( G^{-1} \ Y^t G \right)
\]
where $D_j = \left( \begin{array}{cc} qI_j & 0 \\ 0 & I_{n-j} \end{array} \right)$, $G \in SL_n(\mathbb{Z})/\mathcal{K}_j$, $Y \in \mathcal{Y}$ with $\mathcal{Y}$ the set of matrices
\[
\begin{pmatrix}
U & V \\
V^t & 0
\end{pmatrix}
\] so that \( U \in \mathbb{Z}_q^{j,n} \) varies modulo \( q^2 \), \( V \in \mathbb{Z}_q^{n-j} \) varies modulo \( q \). So

\[
\mathbb{E}_{n-\ell}(\tau)|T_j(q^2) = q^{j(n-1)} \sum_{G,Y} \sum_{M,N} \det(M \left(D_j^{-1}G^{-1} + D_j^{-1}Y^tG + N\right)^{-k})
\]

(\text{where } (M,N) \text{ varies over coprime symmetric pairs with rank}_qe^{M,N} = n-\ell).\

Take a coprime symmetric pair \((M,N)\) with rank\(_qM = n-\ell\). Let \(d_1\) be the rank of the first \(j\) columns of \(M\); using row operations, we can assume \(M = \begin{pmatrix} M_1 & M_2 \\ qM_3 & M_4 \end{pmatrix}\) where \(M_1 = d_1 \times j\) (so rank\(_qM_1 = d_1\)), \(M_4 = d_4 \times (n-j)\) with \(\text{rank}_qM_4 = d_4 = n-\ell - d_1\). Correspondingly, write \(N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}\) where \(N_1\) is \(d_1 \times j\) and \(N_4 = d_4 \times (n-j)\). Take \(r\) so that rank\(_q\begin{pmatrix} M_1 & 0 \\ M_5 & N_5 \end{pmatrix}\) = \(n-d_4-r\); so using row operations, we can assume

\[
(qM_5'M_6'N_5'N_6') = \begin{pmatrix} qM_5 & qM_6 & N_5 & N_6 \\ q^2M_7 & qM_8 & N_7 & qN_8 \end{pmatrix}
\]

where \(M_6, N_6\) are \((\ell-r) \times (n-j)\) and rank\(_q\begin{pmatrix} M_1 & 0 \\ M_5 & N_6 \end{pmatrix}\) = \(n-d_4-r\). Note that since \((M,N) = 1\), we must have rank\(_qN_7 = r\). Then with \(D_{d_1,r} = \begin{pmatrix} qI_{d_1} & 1 \\ I & \frac{1}{q}t_r \end{pmatrix}\),

\[
D_{d_1,r}(M,N) \begin{pmatrix} D_j^{-1} & D_j \end{pmatrix} = \begin{pmatrix} M_1 & qM_2 & q^2N_1 & qN_2 \\ M_3 & M_4 & qN_3 & N_4 \\ M_5 & qM_6 & qN_5 & N_6 \\ M_7 & M_8 & N_7 & N_8 \end{pmatrix}
\]

has \(q\)-rank \(n\). Hence for any \(Y \in \mathcal{Y}_j\),

\[
(M'N') = D_{d_1,r}(M,N) \begin{pmatrix} D_j^{-1} & D_j \end{pmatrix} \begin{pmatrix} G^{-1} & Y^tG \\ -1 & tG \end{pmatrix}
\]

is a coprime symmetric pair with rank\(_qM' = n-\ell + t\) for some \(t \geq 0\). Note that

\[
\det(M'\tau + N')^{-k} = q^{k(d_1-r)} \det(MD_j^{-1}G^{-1} + MD_j^{-1}Y^tG + ND_j^{-1}G)^{-k}.
\]

Similar to the computation in the proof of Theorem 4.1, we have

\[
\chi_{N/q}(M,N) = \chi_{N/q}(D_{d_1,r}M_{d_1}D_j^{-1}, D_{d_1,r}D_j)\chi_{N/q}(M',N').
\]
Reversing, take a coprime pair \((M',N')\) with \(\text{rank}_q M' = n - \ell + t\). We need to count the equivalence classes \(SL_n(\mathbb{Z})(M,N)\) so that
\[
D_{d_1,r}(M,N) \left( \begin{array}{c} D_j^{-1} \\ D_j \end{array} \right) \left( \begin{array}{c} G^{-1} \\ t'G \end{array} \right) \in SL_n(\mathbb{Z})(M',N').
\]

For \(E_1, E_2 \in SL_n(\mathbb{Z})\) and
\[
(M_i N_i) = D_{d_1,r}^{-1} E_i (M',N') \left( \begin{array}{c} G \\ G^{-1} \end{array} \right) \left( \begin{array}{c} D_j \\ D_j \end{array} \right),
\]
we have \((M_1 N_1) \in SL_n(\mathbb{Z})(M_2 N_2)\) if and only if \(E_1 \in K_{d_1,r} E_2\). Thus we need to count the number of triples \(E,G,Y\) with \(E \in K_{d_1,r} \setminus SL_n(\mathbb{Z}), G \in SL_n(\mathbb{Z})/K_j, Y \in \mathcal{Y}_j\) so that
\[
(M N) = D_{d_1,r}^{-1} E(M',N') \left( \begin{array}{c} G \\ G^{-1} \end{array} \right) \left( \begin{array}{c} D_j \\ D_j \end{array} \right)
\]
is an integral coprime pair with \(\text{rank}_q M = n - \ell\) (that \(M' \cdot N\) is symmetric is automatic).

For \(E,G \in SL_n(\mathbb{Z})\), let \((M_1 M_2)\) be the top \(d_1\) rows of \(EM'G\) with \(M_1\) size \(d_1 \times j\); similarly, let \((N_1 N_2)\) be the top \(d_1\) rows of \(EN'G^{-1}\) with \(N_1\) size \(d_1 \times j\). To have \(M\) integral we need \(q|M_2\). To have \(N\) integral, we will need to solve
\[
N_1 \equiv M_1 U + M_2 V (q^2), \quad N_2 \equiv M_1 V (q)
\]
Since \((M',N') = 1\) and \(q|M_2\), we must have \(\text{rank}_q (M_1 N_1 N_2) = d_1\); thus we can only solve the above congruences if \(\text{rank}_q M_1 = d_1\). So suppose we have chosen \(E,G\) to meet this condition; write
\[
EM'G = \left( \begin{array}{c} M_1 \\ M_3 M_4 \\ M_5 M_6 \\ M_7 M_8 \end{array} \right), \quad EN'G^{-1} = \left( \begin{array}{c} N_1 \\ N_3 N_4 \\ N_5 N_6 \\ N_7 N_8 \end{array} \right)
\]
where \(M_1, N_1\) are \(d_1 \times j\), \(M_4, N_4\) are \(d_4 \times (n - j)\), \(M_5, N_5\) are \((n - r - d) \times j\) where \(Y = \left( \begin{array}{c} U \\ V \end{array} \right) \mathcal{Y}_j\). To have \(\text{rank}_q M = n - \ell\), we need to have \(\text{rank}_q \left( \begin{array}{c} M_1 \\ 0 \\ M_4 \\ 0 \end{array} \right) = n - \ell\); so suppose we have chosen \(E,G\) to meet this condition as well. Then, using left multiplication from \(K_{d_1,r}\) and right multiplication from \(K_j\), we can assume \(\text{rank}_q M_4 = d_4 = n - \ell - d_1\) and \(M_6 \equiv 0 (q)\). Now write \(M_5 = (A'_i A_i), N_5 = (B'_i B_i)\) where, for \(i\) odd, \(A'_i, B'_i\) have \(d_1\) columns, and for \(i\) even, \(A'_i, B'_i\) have \(d_1\) columns. By adjusting further using \(K_{d_1,r}\) and \(K_j\), we can assume that \(\text{rank}_q A'_i = d_1, \text{rank}_q A'_i = d_4, A'_i \equiv 0 (q^2)\) for \(i \neq 1, 4, A_1, A_3 \equiv 0 (q)\), and with \(d_i = \text{rank}_q A_i\) for \(i = 5, 7, 8\), we can assume
\[
A_5 \equiv \left( \begin{array}{c} \alpha_5 \\ 0 \\ 0 \end{array} \right) (q^2), \quad A_6 \equiv \left( \begin{array}{c} 0 \\ q \alpha_6 \\ 0 \end{array} \right) (q^2),
\]
So to have \( N - \ell \) rows of \( \alpha_7 \equiv 0 \pmod{q} \) and \( \alpha_8 \equiv 0 \pmod{q} \), we need to choose \( U \) so that \( U + \sum A_i \) is integral. Thus we need to choose \( U \) and \( V \) are congruent modulo \( q \). Similarly, we need to choose \( B \), so that \( B + \sum A_i \) is integral. Thus we need to choose \( B \) and \( V \) are congruent modulo \( q \). With these (unique) choices of \( U \) and \( B \), so we automatically get \( B + \sum A_i \equiv 0 \pmod{q} \). With these choices, the lower \( \ell - r - d_5 \) rows of \( B' \) are \( 0 \pmod{q} \), and the top \( r - d_7 - d_8 \) rows of \( B' \) are \( 0 \pmod{q} \).

Then by symmetry, we have \( \beta_4, \beta_5, \gamma_4, \delta_1, \delta_2, \epsilon_2 \equiv 0 \pmod{q} \), and \( q \) must divide the lower \( \ell - r - d_5 \) rows of \( B' \) and the upper \( r - d_7 - d_8 \) rows of \( B' \).

With \( Y = \begin{pmatrix} U & V \\ t_V & 0 \end{pmatrix} \) (as above), write

\[
U = \begin{pmatrix} U_1 & U_2 \\ tU_3 & U_3 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & V_2 \\ tV_3 & V_3 \end{pmatrix}
\]

where \( U_1 \) is \( d_1 \times d_1 \) and \( V_1 \) is \( d_1 \times d_4 \). To have \( N \) integral, we need

\[
N_1 \equiv A_1' (U_1 U_2) (q^2), \quad N_2 \equiv A_1' (V_1 V_2) (q), \quad B_2 \equiv A_4' \times V_3 \equiv (q).
\]

With these (unique) choices of \( U_1, U_2, V_1, V_2, V_3 \), the symmetry of \( M' \times N' \) implies that

\[
B_3' \equiv A_1' (q), \quad B_2 \equiv A_4' \times V_3 \equiv (q).
\]

so we automatically get \( B_3' \equiv A_1' (q) \). Hence with these choices of \( U_1, U_2, V_1, V_2, V_3 \), the top \( n - \ell \) rows of \( N \) are integral. We have already ensured the top \( n - \ell \) rows of \( M \) are integral with \( q \)-rank \( n - \ell \), and we know the lower \( \ell \) rows of \( M \) are \( 0 \pmod{q} \). So we need to choose \( U_3, V_4 \) so that the lower \( \ell \) rows of \( N \) are integral with \( q \)-rank \( \ell \).

By symmetry, we have

\[
B_3' \equiv A_5' B_1 + A_6' B_2 \equiv A_5' V_2 \times A_1' (q^2), \quad B_6' \equiv A_5' B_3 \equiv A_5 V_3 \times A_4' (q),
\]

so to have \( N \) integral, we need to choose \( E, G \) so that \( \beta_6 \equiv 0 \pmod{q} \), and \( U_3 \equiv A_5 U_3 (q) \). With such choices, the lower \( \ell \) rows of \( N \) are congruent modulo \( q \) to

\[
\begin{pmatrix}
0 & (B_3 - A_5 U_3 - A_6 V_4) / q \\
0 & B_7 - A_7 U_3 - A_8 V_4
\end{pmatrix}.
\]
Also, since \((M', N') = 1\), when \(\beta_0 \equiv 0 (q)\), we will necessarily have \(\text{rank}_q \gamma_3 = \ell - r - d_5\) (recall that \(\beta_3, \beta_5, \gamma_4 \equiv 0 (q)\)). Write

\[
U_3 = \begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 \\
\nu_1 & \nu_2 & \nu_3 \\
\nu_4 & \nu_5 & \nu_6
\end{pmatrix}, \quad V_4 = \begin{pmatrix}
\nu_1 & \nu_2 \\
\nu_3 & \nu_4 \\
\nu_5 & \nu_6
\end{pmatrix}
\]

where \(\mu_1\) is \(d_5 \times d_5\), \(\mu_4\) is \(d_7 \times d_7\), \(\nu_2\) is \(d_5 \times d_6\), and \(\nu_4\) is \(d_7 \times d_8\). Note that

\[
B_7 - A_7 U_3 - A_8 \, ^t V_4 \equiv \begin{pmatrix}
0 & 0 & \delta_3 \\
\delta_4 - \alpha_7^t \mu_2 & \delta_5 - \alpha_7 \mu_4 & \delta_6 - \alpha_7 \mu_5 \\
\delta_7 - \alpha_8^t \nu_2 & \delta_8 - \alpha_8 \nu_4 & \delta_9 - \alpha_8 \nu_6
\end{pmatrix} (q).
\]

So to have

\[
\text{rank}_q \begin{pmatrix}
0 & (B_5 - A_5 U_3 - A_6 \, ^t V_4)/q & 0 & B_6 - A_5 V_4 \\
0 & B_7 - A_7 U_3 - A_8 \, ^t V_4 & 0 & 0
\end{pmatrix},
\]

we need to choose \(E, G\) so that \(\text{rank}_q \delta_3 = r - d_7 - d_8\). We know that \(\gamma_3 = (\ell - r - d_5) \times (n - j - d_4 - d_5)\) and \(\delta_3 = (r - d_7 - d_8) \times (j - d_4 - d_5 - d_7)\). Thus if \(\beta_0 \equiv 0 (q)\) and \(\text{rank}_q \delta_3 = r - d_7 - d_8\), we have

\[
\ell - r - d_5 \leq n - j - d_4 - d_5, \quad r - d_7 - d_8 \leq j - d_4 - d_5 - d_7,
\]

and consequently \(r = j - d_4 - d_5 + d_6\) (recall that \(n - \ell = d_4 + d_5\)). Then we use left multiplication from \(K_j\) to modify \(G\) so that we can assume \(\beta_4 \equiv 0 (q^2)\).

Thus we need to choose \(K_{d_1, r}, E, GK_j\) so that (adjusting the coset representatives \(E, G\)), the top \(d_1\) rows of \(EM'\) have \(q\)-rank \(d_1\), the top \(d_4 + d_5 + d_8\) rows of \(EM'\) have \(q\)-rank \(d_1 + d_4 + d_5\) (where \(0 \leq d_5 \leq j - d_4\)), and \(q\) divides rows \(d_1 + d_4 + d_5 + 1\) through \(n - d_7 - d_8\) of \(EM'\); Lemma? tells us that the number of such \(K_{d_1, r} E\) is

\[
\beta(d', d + d_5) \beta(n - d', n - r - d - d_5) \beta(d + d_5, d_4)
\]

\[
\cdot \gamma^{(d + d_5)(r + d + d_5 - d') + d_4 (n - d - d_5)}
\]

where \(d = \text{rank}_q M, d' = \text{rank}_q M'\) (note that after choosing \(E\) as in the lemma, we can use left multiplication from \(K_{d_1, r}\) to ensure rows \(d_1 + d_4 + d_5 + 1\) through \(n - d_7 - d_8\) are divisible by \(q\)). Then we can choose some \(G_0 \in SL_n(\mathbb{Z})\) so that

\[
EM' G_0 \equiv \begin{pmatrix}
C' & 0 & 0 & 0 \\
0 & C'' & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & C'' & 0
\end{pmatrix} (q)
\]

where \(C\) is \(d_1 \times d_1\) with \(\text{rank}_q C = d_1, C' = (d_4 + d_5) \times (d_4 + d_5)\) with \(\text{rank}_q C' = d_4 + d_5\). As \(GK_j\) varies over \(SL_n(\mathbb{Z})/K_j\), so does \(G_0 GK_j\); Lemma? tells us that the number of \(GK_j\) that meet all the necessary criteria as described above is

\[
\beta(d_4 + d_5, d_4) \beta(d_7 + d_8, d_5) q^{(d_4 + d_5)(j - d_4 - d_5) - d_7 - d_8}.
\]
Having chosen such $E,G$, we have seen that to have $N$ integral, $U_1, U_2, V_1, V_2, V_3$ are uniquely determined, and $\mu_1, \mu_2, \mu_3$ are determined modulo $q$. To also have $(M,N) = 1$, we need to ensure rank$_q B = \ell$ where

$$
B = \begin{pmatrix}
(\beta_1 - \alpha_5 \mu_1)/q & (\beta_2 - \alpha_5 \mu_2)/q & (\beta_3 - \alpha_5 \mu_3)/q & \gamma_1 - \alpha_5 \nu_1 & \gamma_2 - \alpha_5 \nu_2 \\
0 & * & * & \gamma_3 & 0 \\
0 & 0 & \delta_3 & 0 & 0 \\
\delta_7 - \alpha_8 \nu_2 & \delta_8 - \alpha_8 \nu_4 & \delta_9 - \alpha_8 \nu_6 & 0 & 0 \\
\end{pmatrix}.
$$

We have $\delta_3$ square and invertible modulo $q$; so we need $\delta_7 - \alpha_7 \mu_4$ (which is square) to be invertible modulo $q$. By symmetry, we know $(\delta_7 - \alpha_7 \mu_4)^t \alpha_7$ is symmetric; writing $\mu_4 = \mu_4' + q \mu_4''$ where $\mu_4', \mu_4''$ vary over symmetric $d_7 \times d_7$ matrices modulo $q$, $(\delta_7 - \alpha_7 \mu_4)^t \alpha_7$ does as well. (So there are $q^{d_7(2d_7 + 1)/2}$ ways to choose $\mu_4$ so that $\delta_7 - \alpha_7 \mu_4$ is invertible modulo $q$.) So to have $B$ invertible, we need

$$
\begin{pmatrix}
(\beta_1 - \alpha_5 \mu_1)/q & \gamma_1 - \alpha_5 \nu_1 & \gamma_2 - \alpha_5 \nu_2 \\
0 & \gamma_3 & 0 \\
\delta_7 - \alpha_8 \nu_2 & 0 & 0 \\
\end{pmatrix}
$$

to be invertible modulo $q$. We previously noted that $\gamma_3$ is invertible modulo $q$, so we need

$$
\begin{pmatrix}
(\beta_1 - \alpha_5 \mu_1)/q & \gamma_2 - \alpha_5 \nu_2 \\
\delta_7 - \alpha_8 \nu_2 & 0 \\
\end{pmatrix}
$$

to be invertible modulo $q$, or equivalently, we need

$$
\begin{pmatrix}
(\beta_1 - \alpha_5 \mu_1)/q & (\gamma_2 - \alpha_5 \nu_2)^t \alpha_5 \\
(\delta_7 - \alpha_8 \nu_2)^t \alpha_8 & 0 \\
\end{pmatrix}
$$

to be invertible modulo $q$, and this latter matrix is symmetric modulo $q$.

Now we compute $\sum_Y \chi_q(M,N)\chi_q(M',N')$. First, we choose a permutation matrix $G_1 \in GL_n(\mathbb{Z})$ so that

$$
EM'GG_1 \equiv \begin{pmatrix}
A_1' & 0 & 0 & 0 \\
0 & A_4' & 0 & 0 \\
0 & 0 & A_5 & 0 \\
0 & 0 & A_7 & A_8 \\
\end{pmatrix} (q),
$$

$$
EN'^t G^{-1} G_1^{-1} = \begin{pmatrix}
B_1 & B_2 & B_3 & B_4 \\
B_5 & B_6 & B_7 & B_8 \\
B'_1 & B'_2 & B'_3 & B'_4 \\
B'_5 & B'_6 & B'_7 & B'_8 \\
\end{pmatrix}
$$
(recall that since $G_1$ is a permutation matrix, $^tG_1^{-1} = G_1$). Then

$$MG_1 \equiv \begin{pmatrix} A_1' & A_4' \\ & 0 \\ & & 0 \end{pmatrix} (q),$$

$$N^tG_1^{-1} \equiv \begin{pmatrix} * & * & * & * \\ * & * & (B_5 - A_5U_3 - A_6^tV_4)/q & B_6 - A_5V_4 \\ 0 & 0 & B_7 - A_7U_3 - A_8^tV_4 & 0 \end{pmatrix} (q).$$

Then we choose permutation matrices $E_2', G_2' \in GL_{n-d_i-d_{t-4}}(\mathbb{Z})$ so that

$$E_2' \begin{pmatrix} A_5 & 0 \\ A_7 & A_8 \end{pmatrix} G_2' \equiv \begin{pmatrix} \alpha_5 & 0 \\ \alpha_8 & \alpha_7 \\ 0 & \end{pmatrix} (q),$$

$$E_2' \begin{pmatrix} (B_5 - A_5U_3 - A_6^tV_4)/q & B_6 - A_5V_4 \\ B_7 - A_7U_3 - A_8^tV_4 & 0 \end{pmatrix} \equiv \begin{pmatrix} \delta_7 - \alpha_8^t\nu_2 & \gamma_2 - \alpha_8\nu_2 & * & 0 \\ \beta_1 - \alpha_5\mu_1)/q & 0 & 0 & 0 \\ 0 & 0 & \delta_5 - \alpha_7\mu_4 & 0 \\ 0 & 0 & \delta_3 & \end{pmatrix} (q).$$

Set $E_2 = \begin{pmatrix} I_{d_1+d_4} & E_2' \end{pmatrix}$, $G_2 = \begin{pmatrix} I_{d_1+d_4} & G_2' \end{pmatrix}$. Then

$$\chi_q(\det(E_2G_1G_2)) \chi_q(M', N') = \chi_q(E_2EM'GG_1G_2, E_2EN'(GG_1G_2)^{-1})$$

$$= \chi_q(\det A_1' \cdot \det A_4' \cdot \det \alpha_5 \cdot \det \alpha_7 \cdot \det \alpha_8) \chi_q(\det \gamma_3 \cdot \det \delta_3).$$

On the other hand,

$$\chi_q(\det(E_2G_1G_2)) \chi_q(M, N) = \chi_q(E_2MG_1G_2, E_2N'(G_1G_2)^{-1})$$

$$= \chi_q(\det A_1' \cdot \det A_4') \chi_q(\det \gamma_3 \cdot \det \delta_3)$$

$$\cdot \chi_q \left( \det \begin{pmatrix} (\beta_1 - \alpha_5\mu_1)/q & \gamma_2 - \alpha_8\nu_2 \\ \delta_7 - \alpha_8^t\nu_2 & \end{pmatrix} \cdot \det(\delta_5 - \alpha_7\mu_4) \right).$$

Thus

$$\chi_q(M, N) \chi_q(M', N') = \chi_q(\det(\begin{pmatrix} (\beta_1 - \mu_1^t\alpha_5)/q & \gamma_2 - \nu_2^t\alpha_5 \\ \delta_7 - \nu_2^t\alpha_8 & 0 \end{pmatrix} \cdot \det(\delta_5 - \mu_4^t\alpha_7)).$$
are symmetric modulo \(q\). Thus
\[
\sum_{\mu_1, \mu_2} \chi_q \left( \det \left( \frac{\pi \gamma_1 - \mu_1}{\pi \gamma_2 - \mu_2} \right) \det(\pi \delta_5 - \mu_1) \right) = \text{sym}^\chi_q(d_5, d_8),
\]
and
\[
\sum_{\mu_4} \chi_q(\det(\pi \delta_5 - \mu_4)) = \text{sym}^\chi_q(d_7).
\]

We have seen that \(\mu_2, \mu_3\) are determined modulo \(q\), but unconstrained further modulo \(q^2\), \(\mu_5, \mu_6\) are unconstrained modulo \(q^2\), and \(\nu_1, \nu_3, \nu_4, \nu_5, \nu_6\) are unconstrained modulo \(q\). Hence there are \(q(j-d_1)(n-d_1-d_2+1)\) choices for \(Y\) so that \(M, N\) are integral with \((M, N) = 1\). Having fixed \(E, G\) and then summing over those \(Y\) that meet the conditions determined above,
\[
\sum_Y \mathcal{X}(M, N) \chi_q(M', N') = q^{(j-d_1)(n-d_1-d_2+1) - d_5(j-d_1-d_2+1) - d_7(d_5+1)/2} \text{sym}^\chi_q(d_7) \text{sym}^\chi_q(d_5, d_8).
\]

To simplify the formula for \(A_j(d, t)\), we note that \(r = j - d_1 - d_5 + d_8, d = d_1 + d_4 = n - \ell, d' = d + t, t = d_5 + d_7 + d_8, d_1 + d_4 + d_7 \leq j, d_4 + d_8 \leq n - j,\) and \(d_8 \leq d_5\). Using this information yields the formula for \(A_j(\ell; d_1, d_5, d_8)\). Also, we know \(\beta(m, s) = \beta(m, m - s)\), so
\[
\beta(d_1 + d_4 + d_5, d_1) \beta(d_1 + d_4 + d_5, d_4) = \mu(n - \ell + d_5, d_1) \mu(n - \ell + t, t - d_5) \mu(n - \ell - d_1 + d_5, d_5) \mu(t, d_5)
\]
\[
= \mu(n - \ell + d_1 - t, t - d_5) \mu(n - \ell - d_1 + d_5, d_5) \mu(n - d_5, d_5) \mu(t, d_5)
\]
\[
= \beta(d_1 + d_4 + d_5, d_1) \beta(d_1 + d_4 + d_5, d_4).
\]

This gives us the formula for \(A_j(d, t)\), subject to the constraints on the \(d_i\). Taking \(0 \leq d_1 \leq j, 0 \leq d_5 \leq j - d_1,\) and \(0 \leq d_8 \leq d_5,\) the summand in the formula for \(A_j(d, t)\) is 0 if the other constraints on the \(d_i\) are not met. □

As discussed after Theorem ??, we know we have a basis \(\{\overline{E}_p\}_p\) of simultaneous eigenforms for the space of Eisenstein series of degree \(n\), weight \(k\), square-free level \(N\), and character \(\chi\), and these are eigenforms for all Hecke operators \(T(p), T_j(p^2)\) where \(p\) is any prime. Below we compute the eigenvalues for \(T_j(q^2)\) (where, as above, \(q|N\); in later work we compute the eigenvalues for \(T(p), T_j(p^2)\) for \(p\) any prime not dividing \(N\).
Corollary 4.5. Let ρ be a multiplicative partition of N, and suppose \( \mathbb{E}_\rho \neq 0 \). Then with \( d = \text{rank}_q M_\rho \), for a prime \( q|N \) and \( d = \text{rank}_q M_\rho \), we have \( \tilde{\mathbb{E}}_\rho | T_j(q^2) = \lambda_{\rho,j}(q^2) \tilde{\mathbb{E}}_\rho \) where

\[
\lambda_{\rho,j}(q^2) = q^{jd} \sum_{d_1=0}^j q^{d_1(2k-2d-j+d_1-1)} \chi_{\mathcal{N}_0}(q^{2d_1}) \chi_{\mathcal{N}_n}(q^{2(j-d_1)}) \beta(d, d_1) \beta(n-d, j-d_1).
\]

Proof. By Corollary 4.3 and Theorem 4.4, we know that \( \tilde{\mathbb{E}}_\rho \) is an eigenform for \( T_j(q^2) \) with eigenvalue \( A_j(d,0) \). In general, with \( r = j - d_1 - d_5 + d_8 \), and prime \( q'|N/q \) so that \( d' = \text{rank}_{q'} M_\rho \), we know \( \chi_{q'}^2 = 1 \) for \( q' | N/(\mathcal{N}_0 \mathcal{N}_n) \) and thus

\[
\chi_{q'}(D_{d_1,r} M_\rho D_j^{-1}, D_{d_1,r} D_j) = \begin{cases} 
\chi_{q'}'(q^{d_5-d_8}) & \text{if } 0 < d' < n, \\
\chi_{q'}^2(q^{d_1}) \chi_{q'}(q^{d_5-d_8}) & \text{if } d' = 0, \\
\chi_{q'}^2(q^{j-d_1}) \chi_{q'}(q^{-d_5+d_8}) & \text{if } d' = n.
\end{cases}
\]

Since in the sum for \( A_j(d,0) \) we have \( d_5, d_8 = 0 \), the corollary follows. \( \square \)