The Jahn–Teller effect in the presence of partial isotopic substitution: the $\tilde{B}^1E''$ state of NH$_2$D and NHD$_2$†

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Rotationally resolved resonance enhanced multiphoton ionisation spectra of the $\tilde{B}^1E''$ state of NH$_2$D are presented and analysed. The analysis indicates a small (34.9 cm$^{-1}$) lifting of the vibronic degeneracy of the zero point level, approximately equal in sign but opposite in magnitude to the splitting observed in NHD$_2$ in previous work. This observation is consistent with previous measurements on systems with partial isotopic substitution subject to a mild Jahn–Teller effect. A model is developed to calculate the splitting induced by asymmetric isotopic substitution of a degenerate electronic state, based on a harmonic force field with linear and quadratic Jahn–Teller terms added. The force field is developed in internal co-ordinates to allow the same parameters to be used to calculate the pattern of vibronic levels for all four isotopologues. The lifting of the degeneracy of the zero point level on asymmetric substitution comes from the quadratic Jahn–Teller effect; the linear term does not lift the degeneracy.

Introduction

The Jahn–Teller effect is an important consideration in understanding the structure and spectroscopy of degenerate electronic states. If strong (also known as a static Jahn–Teller effect) the geometry of the molecule changes to break the symmetry. We are focussing here on the dynamic Jahn–Teller effect, where the energy shifts are smaller, typically less than the zero point energy, and thus where the symmetry of the molecule is essentially maintained (or at least gives the best starting point for modelling). To interpret spectra additional terms in the effective Hamiltonian are required to account for the Jahn–Teller effect giving shifts and splittings in both the vibrational and rotational energy levels. These terms are reasonably well understood, (see for example a review by Barckholtz and Miller) though examples of full rotational analysis are relatively rare in the literature.

In this paper we look at the additional complications introduced in the presence of partial isotopic substitution. This breaks the symmetry of the vibrational and rotational terms, but must leave the electronic symmetry unbroken. To understand the resulting energy level pattern requires additional terms above the standard vibrational or rotational Hamiltonian and this paper seeks to understand these by a combination of experimental measurements on a reasonably well understood system and theoretical considerations of the simplest linear and quadratic terms in the potential.

The system looked at here is the electronically degenerate $\tilde{B}^1E''$ state of ammonia for which the $\tilde{B}^1E''$–$X^3A_1$ transition gives strong resonance enhanced multiphoton ionization (REMPI) spectra and is a prototypical example of a system with a mild Jahn–Teller effect. It is also an important transition in its own right, as it has become a standard system for probing rotational states of ammonia in the laboratory, as for example in a recent study by Tkac et al. on collisions and in cold molecule work such as that by Twyman et al.

The $\tilde{B}^1E''$ state is planar and the degeneracy arises from the $3p\pi\alpha$ Rydberg orbital in the plane of the molecule, and so the Jahn–Teller effect is relatively weak, giving vibrational energy level shifts of the order of 100 cm$^{-1}$. The rotational structure can be understood in terms of a standard symmetric top with two additional terms in the rotational Hamiltonian ($\zeta$ and $\eta$) arising from a significant residual electronic angular momentum. This electronic angular momentum (almost one unit) also means the state shows a significant Zeeman effect. The rotational and vibrational structure is reasonably well understood for both NH$_2$
and ND₃ but partially isotopically substituted species have not been investigated in any detail. In a previous paper¹ we looked at
resonance enhanced multiphoton ionisation spectra of NH₂D. Formally
this molecule has C₃ᵥ symmetry, and is thus an asymmetric
top with no degenerate vibronic states possible but, as discussed
above, the degeneracy is only lifted at the vibrational and rotational
level. Our analysis of observations indicated that the zero point level
was best understood in terms of two closely spaced vibronic states
with the symmetries A₁ and B₁ that correlate with E" in D₂h. The
separation is only 35.1 cm⁻¹, so the additional terms [ζ and q]
present in the symmetric top Hamiltonian for NH₃ and ND₃ are
required to model the rotational structure, but transformed to
perturbation terms mixing the two states. The terms mixing the
vibronic states are only of the order of magnitude of rotational
terms, but because the states are so close together the spectra cannot
be effectively modelled without them.

There have been few comparable studies in the literature;
perhaps the most closely related is work by Yu et al.⁷ on the X²E_u
state of C₂H₄D and C₂D₄H. These spectra were modelled as simple
asymmetric tops, with the Jahn–Teller terms accounted for using
perturbation theory working through to anomalous values of the
rotational constants. The splittings of the zero point levels were
approximately equal in magnitude (≈9 cm⁻¹) but opposite in sign
for the two isotopologues. A similar pattern of splittings was
observed by Melnik et al.⁸ in the X²E state of CH₂DO and CD₂HO
though the rotational analysis was more complicated than that
considered here due to the presence of significant spin–orbit
coupling. Other related work is discussed below. There has been
correspondingly little theoretical work on asymmetrically isotopically
substituted Jahn–Teller systems though there is one particularly
relevant recent paper by Iwahara et al.¹⁰ This paper showed that
vibronic levels of a degenerate electronic state remain degenerate
upon arbitrary isotopic substitution, even in the presence of a linear
Jahn–Teller effect. For the current work, this implies we must
consider both linear and quadratic Jahn–Teller terms.

In the current paper we present a set of spectra of the B¹E''
state of NH₂D to complement our earlier spectra of NH₂D₂, and
then look at relating the Jahn–Teller effects in all four isotopologues
by developing a simple model for the force field in internal coordinates that can be applied without adjustment to all four species.

**Experimental**

Spectra of the B¹E''→X¹A' transition in NH₂D were recorded using
resonance enhanced multiphoton ionization spectroscopy (REMPI)
in a molecular beam, using the method described previously.⁶ This
gave rotationally resolved spectra with an effective rotational tem-
perature of 3–90 K, that could be varied by adjusting the expansion
conditions.

**Rotational analysis**

The spectra taken covered 7 members of the bending progres-
sion (ν₂), starting with the origin band. Typical spectra of the
first 6 bands are shown in Fig. 1. All the bands showed clear
rotational structure, and the observed linewidths (~0.9 cm⁻¹)
are probably limited by the laser system, and there was no
indication of rotationally or vibrationally dependent predissocia-
tion at this resolution.

The theory required for the rotational analysis is developed in
our previous paper⁶ and is only briefly summarised here. The basic
model is two distinct vibronic states, for each of which a
standard asymmetric top Hamiltonian in a IIIr representation
is used as a starting point:

\[ \hat{H}_{\text{rot}} = \frac{3}{2}(A + B)(\hat{J}^2 - \hat{J}_z^2) + \frac{3}{2}(A - B)(\hat{J}_x^2 + \hat{J}_y^2) + C\hat{J}_z^2 \]

For the purposes of this paper |+⟩ and |−⟩ are used to label the
two vibronic states, and two additional terms in the Hamiltonian
are required to take account of the residual electronic angular
momentum, both of which introduce terms acting between the
two states. The largest of these is the equivalent of the Coriolis
term in symmetric tops:

\[ \hat{H}_{\text{cor}} = -2C\hat{J}_z \hat{L}_z \]

where we have introduced an effective electronic angular
momentum operator that has electronic matrix elements:

\[ \langle ±|\hat{L}_z|± \rangle = 0, \quad \langle ±|\hat{L}_z|± \rangle = \zeta \]

The overall matrix elements are therefore −2CζK. The other
term is analogous to I doubling in symmetric tops, and can be
written as:

\[ \hat{H}_I = \frac{1}{2}i\hat{q}(\hat{J}_x^2 - \hat{J}_y^2)(\hat{L}_z^2 - \hat{L}_z^2) \]

where \( \hat{J}_x^2 - \hat{J}_y^2 = 2i(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x) \) and the electronic operator
\( \hat{L}_z^2 - \hat{L}_z^2 \) is discussed below. This has matrix elements:

\[ \langle +, J, K | -| \hat{H}_I | -, J, K \rangle = -\frac{1}{2}i\hat{q}\sqrt{\{J(J+1)-(K-1)(K+1)\}\{J(J+1)-K(K-1)\}} \]

\[ \langle +, J, K + 2 | \hat{H}_I | -| J, K \rangle = +\frac{1}{2}i\hat{q}\sqrt{\{J(J+1)-(K+1)(K+2)\}\{J(J+1)-K(K+1)\}} \]

Note that the matrix elements are imaginary, and it is not
possible to make a simple adjustment in phase to make all
matrix elements real. The equations above are used to calculate
the energy levels of the excited state by setting up and diagonalising a (complex) Hamiltonian matrix containing, for a given J
and vibronic symmetry, all possible values of K for both
vibronic components. To complete the model equal two photon
transition moments from the ground state to each of the two
excited states is assumed and, as discussed in the previous
paper, interference between these two has a significant effect
on the pattern of observed transitions. The calculations were
carried out using the PGOPHER program.¹¹,¹² The ground state
constants were taken from Fusina et al.¹³
The $\tilde{B}^1E'' \rightarrow \tilde{X}^1A'$ transition involves a change in geometry from pyramidal to planar, so the transition shows a progression in $\nu_2$, the umbrella mode. In the upper state members of this progression alternate between $E^0$ and $E'$ symmetry for even and odd $\nu_2$ respectively, and the selection rules mean that the lower state of the electronic transition alternates between the upper and lower tunnelling component in the ground state. As this results from levels symmetric and antisymmetric with respect to the plane of symmetry of the molecule, which persists for NH$_2$D and NHD$_2$, the same pattern of transitions is observed.

Fig. 1 MPI spectra of the $\tilde{B}^1E'' \rightarrow \tilde{X}^1A'$ transition in NH$_2$D. In each panel, the upper trace is the experimental spectrum and the lower trace is a simulation using the Hamiltonian and parameters described in the text.
for all four isoplogues. In NH₂D the |±⟩ levels alternate between A₂ (even v₂) and B₁ (odd v₂) and the |∓⟩ levels alternate between B₁ and A₂ vibronic symmetry.

Given this model the assignment of the observed spectra was reasonably straightforward, given the guidance of the fits for NHD₂. Table 1 shows four alternative fits to the pair of states making up the origin band. These follow the pattern in the previous paper; fit I is the simplest model, assuming two non-interacting asymmetric top states. This gives a reasonable fit, but is clearly worse than the other models which allow for interaction between the states. Fit IV is the most flexible model, adding the two interactions discussed above and allowing A, B and C for both components to float independently. This is clearly a better fit than fit I as the average error drops by a factor of two from 0.53 cm⁻¹ independently. This is clearly a better fit than fit I as the average error drops by a factor of two from 0.53 cm⁻¹. This is clearly a better fit than fit I as the average error drops by a factor of two from 0.53 cm⁻¹.

Table 1 Rotational constants* from alternative fits to the rotational structure of the origin band of the B†E'' - X̃Ã₁ transition in NH₂D

<table>
<thead>
<tr>
<th>Origin</th>
<th>Fit I</th>
<th>Fit II</th>
<th>Fit III</th>
<th>Fit IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₂</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Estimated</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A₂</td>
<td>59259.78(20)</td>
<td>59294.29(24)</td>
<td>59295.58(10)</td>
<td>59294.44(10)</td>
</tr>
<tr>
<td>B₁</td>
<td>6.51</td>
<td>6.51</td>
<td>6.51</td>
<td>6.51</td>
</tr>
<tr>
<td>q</td>
<td>0.53</td>
<td>0.53</td>
<td>0.53</td>
<td>0.53</td>
</tr>
<tr>
<td>n̂_obs</td>
<td>28</td>
<td>28</td>
<td>28</td>
<td>28</td>
</tr>
</tbody>
</table>

* Units are cm⁻¹; figures in brackets are standard deviations in units of the least significant figure.

Table 2 Rotational constants* for the B†E'' - X̃Ã₁ transition in NH₂D

<table>
<thead>
<tr>
<th>v₂ = 0</th>
<th>v₂ = 1</th>
<th>v₂ = 2</th>
<th>v₂ = 3</th>
<th>v₂ = 4</th>
<th>v₂ = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₂</td>
<td>59259.78(20)</td>
<td>60092.55(72)</td>
<td>60957.62(80)</td>
<td>61848.198(96)</td>
<td>62759.25(15)</td>
</tr>
<tr>
<td>B₁</td>
<td>6.51</td>
<td>6.51</td>
<td>6.51</td>
<td>6.51</td>
<td>6.51</td>
</tr>
<tr>
<td>n̂_obs</td>
<td>28</td>
<td>22</td>
<td>26</td>
<td>24</td>
<td>28</td>
</tr>
</tbody>
</table>

* Figures in brackets are standard deviations in units of the least significant figure.
for the origin band close to the value predicted from a simple rigid rotor model. The splitting between the two components of the origin band is very close for the two states (34.86 cm\(^{-1}\) for NH\(_2\))D against 35.11 cm\(^{-1}\) for NHD\(_2\)) but opposite in sign, with the \(A\) component at lower energy for NH\(_2\))D and the \(B\) component lower for NHD\(_2\). The similar pattern of results gives further confirmation of the rotational analysis for these two species.

### Vibronic analysis

#### The Jahn–Teller effect in NH\(_3\)

As indicated in the introduction the observation of equal and opposite splittings in the zero point level has been seen in other systems, so it is instructive to develop a model of the force field including the Jahn–Teller effect that can be used for all four isotopologues to explain this splitting. The \(\tilde{B}^1\)\(E^\prime\) state of NH\(_3\) has two Jahn–Teller active modes, the asymmetric stretch, \(\nu_3\) and the in plane bend, \(\nu_4\), both with \(E^\prime\) symmetry. The other two modes are the symmetric \(A(\nu_1^0)^2\) stretch, \(\nu_1\) and the umbrella mode, \(\nu_2\) \((A(\nu_2^0)^2)\). The most complete information so far on the electronic part can be derived by simple substitution:

\[
\begin{align*}
\hat{H}_{\text{JT}} &= \hbar \omega_{\text{JT}} (\hat{q}_x \hat{L}_x^2 + \hat{q}_y \hat{L}_y^2) \\
\hat{H}_{\text{JT}} &= \hbar \omega_{\text{JT}} (\sigma \hat{x} \hat{L}_x^2 + \sigma \hat{y} \hat{L}_y^2) \\
\end{align*}
\]

Dimensionless normal coordinates, \(q_i\), are used here. The \(\hat{L}_i^2\) operators are taken to connect the components of the degenerate electronic state, which we label with \(A = \pm 1\), approximately the electronic angular momentum of the states:

\[
\hat{L}_i^2 \pm 1 = |\mp 1\rangle
\]

The initial analysis of Allen et al.\(^4\) has \(\omega_{\text{JT}} = 3314.1 \text{ cm}^{-1}\), \(|\sigma| = 0.165\), \(\omega_{\text{JT}} = 1625.5 \text{ cm}^{-1}\), \(|k| = 0.1905\) which suggests that the linear term is significant for both \(\nu_3\) and \(\nu_4\). These parameters did not fully account for the observed levels, and an alternative analysis was presented by Allen et al.\(^4\) that accounted for more of the observed levels. This added an additional electronic state to the Hamiltonian, and a term linear in \(q_i\), mixing this with the \(B\) state, though a similar fit was also possible using a quadratic Jahn–Teller term of the form:

\[
\hat{H}_{\text{JT2}} = \hbar \omega_{\text{JT}} (\hat{q}_x \hat{L}_x^2 + \hat{q}_y \hat{L}_y^2) + \hbar^2 \omega_{\text{JT}} (\hat{q}_x \hat{L}_x^2 + \hat{q}_y \hat{L}_y^2)\]

At the level of second order perturbation theory the two are equivalent, and we will use the latter approach here for simplicity.

#### Converting to a \(C_{2v}\) picture

The Jahn–Teller terms above are expressed in terms of operators appropriate to \(D_{\text{inh}}\) symmetry, but to apply these to the unsymmetrical isotopologues we must convert everything to states and operators that have the correct symmetry in \(C_{2v}\) as well as \(D_{\text{inh}}\). For the electronic part the required transformation was discussed in the previous paper:

\[
|\pm\rangle = \frac{1}{\sqrt{2}} (|A = 1\rangle \pm |A = -1\rangle)
\]

The \(|\pm\rangle\) states introduced above in the rotational analysis essentially transform as \(x\) and \(y\), as opposed to the \(A = \pm 1\) states which transform as \(x \pm iy\). For the Jahn–Teller term we require the matrix elements of \(\hat{L}_x^2 \pm \hat{L}_y^2\), which can be derived by simple substitution:

\[
(\pm |L_x^2 \pm L_y^2| \pm 1) = \mp 1\] and \(\langle \pm |L_x^2 \pm L_y^2| \mp 1\rangle = \mp 1\)

The other combinations of operators and signs give zero matrix elements. For the vibrational part we re-express the operator for a single Jahn–Teller mode in terms of Cartesian style components:

\[
\hat{q}_z = \hat{q} \pm i\hat{q}_y
\]

which makes the linear Jahn–Teller term become:

\[
\hat{H}_{\text{JT}} = \hbar \omega_{\text{JT}} (\hat{q}_x (L_x^2 + L_y^2) + \hat{q}_y (L_x^2 - L_y^2))
\]
This is, of course equivalent to the earlier expression in full symmetry, but in this form the symmetry requirements of the \( \hat{q}_x \) and \( \hat{q}_y \) coordinates become clear. The term multiplying \( \hat{q}_x \) is diagonal in the \( |\pm\rangle \) basis, and must therefore correspond to a totally symmetric co-ordinate in \( C_{2v} \) symmetry. It has the effect of adding or subtracting a simple linear term along the \( x \) normal co-ordinate. The term multiplying \( \hat{q}_y \) is purely off-diagonal in the \( |\pm\rangle \) basis, and must therefore have the symmetry of the product of the two electronic components. From the previous paper \(|\pm \rangle \) has \( A_2 \) symmetry and \(|-\rangle \) has \( B_1 \) symmetry for the electronic origin (which has \( E'' \) symmetry in \( D_{3h} \)), requiring \( \hat{q}_y \) to have \( B_2 \) symmetry so that the overall operator, \( i\hat{g}_y(\hat{L}_x^2 - \hat{L}_y^2) \) is totally symmetric. Note that \( i\hat{g}_y(\hat{L}_x^2 - \hat{L}_y^2) \) will give purely imaginary matrix elements, but we can avoid this complication by multiplying one of the electronic basis functions by \( i \).

Applying the same process to the quadratic Jahn–Teller effect yields a more complicated result:

\[
\hat{H}_{\text{JT}} = \frac{g \omega}{2} [i\hat{g}_y(\hat{q}_x^2 - \hat{q}_y^2)(\hat{L}_x^2 + \hat{L}_y^2) - 2i\hat{q}_x\hat{q}_y(\hat{L}_x^2 - \hat{L}_y^2)]
\] (14)

Note that again the matrix element off-diagonal in electronic state are purely imaginary, so we can maintain purely real functions by applying the same phase factor as for the linear Jahn–Teller effect.

**Modelling the Jahn–Teller effect in a Cartesian basis**

Given the above we can exactly reproduce the Jahn–Teller energy level pattern in a Cartesian basis consisting of (harmonic) vibrations along \( x \) and \( y \) with associated quantum numbers \( \nu_x \) and \( \nu_y \) rather than the two dimensional oscillator quantum numbers \( (\nu \text{ and } \ell) \) that would more conventionally be used. Similarly the electronic basis can use the \( |\pm\rangle \) functions mentioned above. Expressed in this format the Hamiltonian contains three types of terms: firstly the conventional harmonic oscillator terms for each vibration (identical format the Hamiltonian contains three types of terms: firstly the

\[\hat{H}_{\text{harm}} = \frac{\hbar \omega}{2} (\hat{q}_x^2 + \hat{p}_x^2 + \hat{q}_y^2 + \hat{p}_y^2)\] (15)

diagonal Jahn–Teller term:

\[\langle \pm | \hat{H}_{\text{JT}} | \pm \rangle = \pm k \omega \alpha \hat{q}_x \] (16)

and a Jahn–Teller term connecting the two electronic states:

\[\langle \mp | \hat{H}_{\text{JT}} | \pm \rangle = k \omega \alpha \hat{q}_y\] (17)

Note that, as suggested above, a phase factor of \( i \) has been applied to \( | - \rangle \) to make the above real. While not immediately obvious, all the eigenvalues of this Hamiltonian are doubly degenerate; see the supplementary notes for an explicit proof of this (ESI†). Including the quadratic terms gives a more complicated picture, which is discussed further below.

This picture can now be converted to the \( C_{2v} \) symmetry required for \( NH_3D \) and \( ND_3H \). A major change is that the frequencies associated with \( \hat{q}_x \) and \( \hat{q}_y \) are now different. The requirement that they have different symmetry in \( C_{2v} \), introduced above means that they will correspond to different normal modes in the lower symmetry. (The specific forms for \( NH_3 \) are discussed below, but one of the obvious separations corresponds to the division into \( N-H \) and \( N-D \) stretches.) The standard harmonic part must now have two distinct vibrational frequencies, which we denote by with \( \omega_x \) and \( \omega_y \):

\[\hat{H}_{\text{harm}} = \frac{\hbar \omega_x}{2} (\hat{q}_x^2 + \hat{p}_x^2) + \frac{\hbar \omega_y}{2} (\hat{q}_y^2 + \hat{p}_y^2)\] (18)

Similarly, two constants are now required to specify the linear Jahn–Teller effect, as there is no longer any requirement that the linear potential terms along the two normal modes are the same, but the operators involved are otherwise the same:

\[\langle \pm | \hat{H}_{\text{JT}} | \pm \rangle = \pm k \omega \alpha \hat{q}_x\] (19)

\[\langle \mp | \hat{H}_{\text{JT}} | \pm \rangle = k \omega \alpha \hat{q}_y\] (20)

The relationships \( | \hat{H}_{\text{JT}} | \mp \rangle = -| \hat{H}_{\text{JT}} | \pm \rangle \) and \( | \hat{H}_{\text{JT}} | \mp \rangle = -| \hat{H}_{\text{JT}} | \mp \rangle \) persist as both components of the electronic state have the same transformation between internal and normal co-ordinates, only differing in the Jahn–Teller terms, so the sign change in eqn (16) is simply copied across. Interestingly, all levels remain doubly degenerate in the presence of the linear Jahn–Teller effect, even including the term (20) that mixes the two electronic states. This has been formally proved by Iwahara et al.,10 and can also be seen on inspecting the detailed form of the Hamiltonian matrices – see the supplementary notes (ESI†).

The above is in fact an oversimplification; the transformation between internal and normal co-ordinates will introduce Jahn–Teller terms to modes that do not have them for the symmetrical isotopologues. The only requirement is that they have the same symmetry in the sub-group. \( \hat{q}_x \) is necessarily totally symmetric (as it produces terms diagonal in electronic state), so any totally symmetric mode can acquire a Jahn–Teller term diagonal in electronic state. Similarly any vibrations with the same symmetry as \( \hat{q}_x \) can acquire a Jahn–Teller term off-diagonal in electronic state. For ammonia this mode mixing is worked out in detail below, and indeed occurs for the symmetric stretch, though there is no vibration with the same symmetry as \( \hat{q}_y \) so no additional off-diagonal terms are introduced. Importantly, though these result in one or more additional Jahn–Teller terms, the same logic still holds in that two identical matrices are generated and the degeneracy is not lifted by any linear terms. This degeneracy can even persist if the symmetry is lowered so that the \( \hat{q}_x \) and \( \hat{q}_y \) modes have the same symmetry, which is only possible for ammonia using a third isotope, say tritium to give an isotopologue with three different isotopes, \( NHDT \). This is the result of Iwahara et al.,10 and was confirmed by numerical tests with the model below.

A purely quadratic Jahn–Teller effect gives rather a different picture. The first point to note is that, even in \( D_{3h} \) symmetry, some non-degenerate vibronic states are present, and the degeneracies reflects the vibronic symmetry. For example, the overall vibronic symmetry of states with one quantum of a degenerate mode excited is \( E'' \times E' = A''_1 + A''_2 + E'', \) and three levels are indeed seen. (If only linear terms are included then the \( A''_1 \) and \( A''_2 \) levels remain degenerate). In addition, the selection rules for the \( q^2 \) operators of \( \Delta \nu = 0, \pm 2 \) means the quadratic terms give rise to effects in first order. This case is worked
through in more detail in the supplementary notes (ESI†). Symmetry requires that the vibrationless level in $D_{sh}$ is degenerate.

For the partially substituted species the quadratic Jahn–Teller Hamiltonian includes terms diagonal in the electronic states:

$$\langle \pm | \hat{H}_{\text{JT}} | \pm \rangle = \pm \frac{1}{2} \left( g_{xx} \delta x^2 - g_{yy} \delta y^2 \right) = \pm \frac{1}{2} \left( g_{xx} \delta x^2 - g_{yy} \delta y^2 \right)$$

and off-diagonal:

$$\langle \pm | \hat{H}_{\text{JT}} | \mp \rangle = -g_{xy} \delta x \delta y$$

(22)

We have introduced a slightly different notation here to simplify later equations. The first term is likely to be the most important, as it can give diagonal matrix elements in selected states. Specifically, first order shifts in opposite directions to the vibrationless levels arise, giving an overall splitting of the zero point level of $\frac{1}{2} (g_{xx} - g_{yy})$, and these will scale with $v$, giving effective harmonic frequencies of approximately $\omega_x \pm g_{xx}/4$ and $\omega_y \pm g_{yy}/4$. The second term is likely to become important for excited state splittings, as can be seen by considering the matrix for one quantum in the degenerate mode:

$$\begin{pmatrix}
| + 10 \rangle \\
| - 01 \rangle
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{2} \omega_x + \frac{1}{2} \omega_y + \frac{g_{xx}}{2} - \frac{g_{yy}}{2} & -g_{xy} \\
-g_{xy} & \frac{1}{2} \omega_x + \frac{1}{2} \omega_y - \frac{g_{xx}}{2} + \frac{g_{yy}}{2}
\end{pmatrix}$$

(23)

$$\begin{pmatrix}
| + 01 \rangle \\
| - 10 \rangle
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{2} \omega_x + \frac{1}{2} \omega_y + \frac{g_{xx}}{2} - \frac{g_{yy}}{2} & -g_{xy} \\
-g_{xy} & \frac{1}{2} \omega_x + \frac{1}{2} \omega_y - \frac{g_{xx}}{2} + \frac{g_{yy}}{2}
\end{pmatrix}$$

(24)

From this it is clear that in addition to the first order splittings (of order $g_{xx} - g_{yy}$) there will be second order shifts of the order of $g_{xy}^2 (g_{xx} - g_{yy})$, which are likely to be significant as all three constants will be the same order of magnitude. As for the linear case, mode mixing will introduce additional potential terms, but all the degeneracies are lifted without them.

**Force constant analysis for the $\tilde{B}^1E''$ state of NH₃**

To translate the Jahn–Teller effect to a change in geometry, and to transfer between the various isotopically substituted species requires knowledge of the internal co-ordinates for NH₃ so we can derive a potential that is independent of mass. For the purposes of this analysis we choose the simplest possible form for the potential, expressed in internal valence coordinates:

$$V = \frac{1}{2} k_{\text{stretch}} (\delta r_1^2 + \delta r_2^2 + \delta r_3^2) + \frac{1}{2} k_{\text{bend}} (\delta \theta_1^2 + \delta \theta_2^2 + \delta \theta_3^2) + \frac{1}{2} k_{\text{umb}} \delta \phi^2$$

(25)

$\delta r_1$ is the change in bond length N–H from the equilibrium value. $\delta \theta_i$ is the change in the HNH bond angle not including bond N–H$_i$. $\delta \phi$ describes the out of plane bending motion; formally it is defined as the vector triple product of unit vectors along the N–H bonds as used by Hedberg and Mills. To will ignore coupling between internal coordinates, and anharmonicity, as this is a reasonable match to the available information, and should be sufficient to understand the pattern of energy levels. Numerical tests with off-diagonal internal force constant terms (such as $k_{\text{bend}} \delta \theta_1 \delta \theta_1$) set to ground state values confirm that, while there are shifts in levels there is no change in the overall pattern. A conventional harmonic force field analysis was set up in POGOPHER$^{11,12}$ based on the above; taking $\nu_3$ and $\nu_4$ as given above for NH₂ and $\nu_3$ as 898.2 cm$^{-1}$ (the 1–0 gap) and the rotational constants from the zero point level gives the parameters shown in Table 4.

There is limited redundancy here given the lack of a Jahn–Teller analysis on ND₃, but a test is available for the symmetric stretch, $\nu_1$. This is not known for the B state, but the value for the $C'$ state, which is expected to have similar bonding, is known to be$^{11}$ 3217.0 cm$^{-1}$. The above constants predict 3143.7 cm$^{-1}$, reasonable agreement considering anharmonicity and coupling between internal coordinates are being ignored.

This force field analysis can be used to calculate the magnitude of the Jahn–Teller distortions; this is worked through in detail in the supplementary notes (ESI†), and indicates bond length changes $\sim 0.014$ Å and bond angle changes of the order of 2.2°. This provides some insight into the source of the difference between the A and B constants for the two electronic states; a simple static picture would predict slightly different geometries for the two electronic states, with a corresponding change in rotational geometries. Changing one of the bond angles by 2.2° as suggested above in fact gives a slightly larger change in rotational constants than the observed difference between the A and B values for the two states; the change of 0.014 Å bond length has rather a smaller effect. Given the Jahn–Teller distortion is dynamic, rather than static, we would not expect this to give a quantitative prediction, but it does suggest the shift comes from the Jahn–Teller effect.

**Descent in symmetry**

To apply the above to the mixed isotopologues we must specify symmetry co-ordinates that are appropriate for both $D_{sh}$ and $C_{2v}$. From the character tables we can draw up the correlations shown in Tables 5 and 6 between the operators and symmetries for the two. Following the previous paper the $C_{3v}$ axis is taken as $x$ in $C_{2v}$ and the out of plane axis is $z$. This means that functions with $B_1$ symmetry are anti-symmetric with respect to reflections in the plane of the molecule.

<table>
<thead>
<tr>
<th>$D_{sh}$</th>
<th>$E$</th>
<th>$C_2$</th>
<th>$\sigma_h$</th>
<th>$\sigma_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{2v}$(conventional)</td>
<td>$E$</td>
<td>$C_2(z)$</td>
<td>$\sigma(zy)$</td>
<td>$\sigma(zy)$</td>
</tr>
<tr>
<td>$C_{2v}$(axis choice used here)</td>
<td>$E$</td>
<td>$C_2(x)$</td>
<td>$\sigma(xy)$</td>
<td>$\sigma(xy)$</td>
</tr>
</tbody>
</table>
Similarly, for the degenerate vibrations we can assign separate symmetries in $C_{2v}$ to the two components, if we make the right choice of coordinates that gives co-ordinates with well-defined symmetries in $C_{2v}$. A possible choice, not unusual for systems with a $C_3$ axis, for the stretching co-ordinates is:

$$S_{3x} = \frac{1}{\sqrt{6}}(2\delta r_1 - \delta r_2 - \delta r_3)$$

$$S_{3y} = \frac{1}{\sqrt{2}}(\delta r_2 - \delta r_3)$$

With this choice of co-ordinates, shown in Fig. 2, and N-H$_1$ aligned along the x axis $S_{3x}$ has $A_1$ symmetry and $S_{3y}$ has $B_2$ symmetry. $H_1$ is required to be the unique atom in NH$_2$D and ND$_3$H to maintain the required symmetry. We make a similar choice for the bending co-ordinates, replacing $\delta r_i$ with $\delta \theta_i$:

$$S_{4x} = \frac{1}{\sqrt{2}}(2\delta \theta_1 - \delta \theta_2 - \delta \theta_3)$$

$$S_{4y} = \frac{1}{\sqrt{2}}(\delta \theta_2 - \delta \theta_3)$$

For consistent symmetry properties $\delta \theta_i$ needs to be the change in the angle between the two bonds not involved in $\delta r_i$. For completeness, the non-degenerate symmetry co-ordinates are:

$$S_1 = \frac{1}{\sqrt{3}}(\delta r_1 + \delta r_2 + \delta r_3) \quad (A_1 \text{ symmetry})$$

$$S_2 = \delta \phi \quad (A_1'' \text{ symmetry})$$

The transformation between dimensionless normal coordinates and symmetry coordinates is given in the supplementary notes (ESI†); we simply note here that they are essentially proportional, with a little mixing between the degenerate modes.

Jahn–Teller terms in internal coordinates

The force field analysis also makes it possible to predict the pattern of vibronic energy levels for each of the isotopologues. For this the Jahn–Teller terms need to be converted from (isotope dependent) dimensionless normal coordinates ($k_i\omega_i$ and $k_i\omega_a$) as used in (13) above to (isotope independent) internal coordinates. There is no unique way of doing this, particularly as the internal coordinates are redundant, but given that the degenerate stretching and bending modes are reasonably separate, we can take the form for the linear terms as:

$$\hat{H}_\text{JT1} = k_{\text{JT1}} \left\{ \frac{1}{\sqrt{6}}(2\delta r_1 - \delta r_2 - \delta r_3)(\hat{L}_x^2 + \hat{L}_y^2) \right\}$$

with a similar term for $q_4$. This is the analogue of eqn (13). In terms of internal valence coordinates this becomes:

$$\hat{H}_\text{JT1} = k_{\text{JT1}} \left\{ \frac{1}{\sqrt{6}}(2\delta r_1 - \delta r_2 - \delta r_3)(\hat{L}_x^2 + \hat{L}_y^2) + i\frac{1}{\sqrt{2}}(\delta r_2 - \delta r_3)(\hat{L}_x^2 + \hat{L}_y^2) \right\}$$

The quadratic terms, eqn (14) become:

$$\hat{H}_\text{JT2} = \frac{1}{2}k_{\text{JT2}} \left\{ (\hat{S}_{3y}^2 - \hat{S}_{3y}^2)(\hat{L}_x^2 + \hat{L}_y^2) - 2i\hat{S}_x\hat{S}_y(L_z^2 - L_{-z}^2) \right\}$$

This has a rather complicated form when expanded out into internal co-ordinates – see the supplementary notes (ESI†). To use the above forms, the harmonic force field part of the PGOPHER11,12 program was enhanced to take additional linear and quadratic potential terms expressed in internal coordinates, and subject them to the transformation between internal coordinates, $R_i$, and dimensionless normal co-ordinates, $q_i$, computed as part of the standard force field analysis. Specifically, if we define the matrix relating the two as $d^{\text{int}}$:

$$R_i = \sum_j d^{\text{int}}_{ij} q_j$$

($d^{\text{int}}$ is given for NH$_3$ in the supplementary notes, ESI†) then the potential energy expressed as a sum of terms linear in internal coordinates as:

$$V = \sum_i \epsilon^{(1)} R_i$$

can easily be recast in terms of normal coordinates by a simple matrix multiplication:

$$V = \sum_i \epsilon^{(1)} R_i = \sum_i \sum_j \sum_l d^{\text{int}}_{ij} q_l = \sum_i \left( \sum_j \epsilon^{(1)} d^{\text{int}}_{ij} \right) q_j$$

so $k_i\omega_j$ can be identified with elements of $\epsilon^{(1)}d^{\text{int}}$. Quadratic terms require a double multiplication:

$$V = \frac{1}{2} \sum_{i,j} \epsilon^{(1)} R_i R_j = \frac{1}{2} \sum_{i,j} \left( \epsilon^{(2)} d^{\text{int}}_{ij} d^{\text{int}} \right) q_i q_j$$

(constant terms require a single multiplication:

$$V = \frac{1}{2} \sum_{i} \left( \sum_l \epsilon^{(1)} d^{\text{int}}_{ij} q_l \right) q_j$$

Table 6 Correlation of symmetries from $D_{3h}$ to $C_{2v}$

<table>
<thead>
<tr>
<th>$D_{3h}$</th>
<th>$A_1'$</th>
<th>$A_2'$</th>
<th>$E'$</th>
<th>$A_1''$</th>
<th>$A_2''$</th>
<th>$E''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{2v}$</td>
<td>$\nu_1$</td>
<td>$\nu_2$</td>
<td>$\nu_3$</td>
<td>$\nu_4$</td>
<td>$\nu_5$</td>
<td>$\nu_6$</td>
</tr>
</tbody>
</table>

| $C_3$ | $A'$ | $A'$ | $A'$ | $A'$ | $A'$ | $A'$ |

Fig. 2 Co-ordinate system used. H$_1$ is D in NH$_2$D and H in ND$_3$H. The arrows show the $S_{3x}$ motion.
and elements of $d^{\text{int}} f^{\text{int}}$ are thus required. The key additional feature in these transformations (not shown in the equations above) is that separate sets of force constants must be tracked, with one set for each of the possible associated electronic matrix elements. For example, for the linear terms, the multipliers of $\hat{L}_2^2 + \hat{L}_2^2$ and $\hat{L}_3^2 - \hat{L}_3^2$ must be kept separate and a separate set of multipliers is required for each state. In PGOPHER these are used to calculate perturbation terms in the vibronic Hamiltonian that are added to the standard harmonic terms.

To determine the required parameters a fit was performed to selected data from Allen et al. The fit was directly to internal coordinate force constants – the conventional $k_{\text{stretch}}$, $k_{\text{bend}}$ and $k_{\text{umb}}$ and the Jahn–Teller terms $k_{\text{JT}3}$, $k_{\text{JT}4}$, and $g_{\text{JT}3}$. A harmonic force field calculation based on the first three gives the harmonic frequencies for the B state, and the transformation matrix, $d^{\text{int}}$. The latter is used to transform the Jahn–Teller terms to the equivalent normal coordinates, it is now straightforward to repeat the process using numerical derivatives by the PGOPHER program is available in many places – see for example Papousˇek and Aliev. The Hamiltonian matrix is diagonalised to produce energies to compare with the observed values above. A standard least squares fitting process using numerical derivatives by the PGOPHER program is used to produce the best fit values listed in Table 7 below, with the observed and calculated values given in Table S2 of the supplementary notes (ESI†).

Note that only the quadratic Jahn–Teller term associated with $\nu_4$ was floated ($g_{\text{JT}4}$); floating the equivalent term for $\nu_4$ did not improve the fit, probably because no $A$ symmetry levels are known for $\nu_4$. The only redundancy present in the fit is in the $\nu_4$ levels, so we do not have an independent measure of the reliability of the derived parameters.

The normal coordinate parameters derived from these are given in the top rows of Tables 8 to 11. Note that, because of the small mixing between $S_3$ and $S_4$, a small quadratic Jahn–Teller term is present for $\nu_4$, even though we are taking the $g_{\text{JT}4}$ term as zero. This mixing also induces a small quadratic cross term with the equivalent operator forms coming out of the $d^{\text{int}} f^{\text{int}}$ transformation a fairly obvious generalisation of eqn (16) and (17):

$$\langle \pm |\hat{H}_{\text{JT}}^{(3)}| \pm \rangle = \pm g_{\text{JT}3}(\hat{q}_{3x}^2\hat{q}_{4x}^2 - \hat{q}_{3y}\hat{q}_{4y}) = \pm (g_{\text{JT}3}^x\hat{q}_{3x}\hat{q}_{4x} + g_{\text{JT}3}^y\hat{q}_{3y}\hat{q}_{4y})$$

(39)

We have introduced a separate quantity, $g_{\text{JT}3}^y$ which is simply the multiplier of the operator $\hat{q}_{3y}\hat{q}_{4y}$; in this case $g_{\text{JT}3}^y = -g_{\text{JT}3}^x$ but this notation simplifies the presentation for the less symmetrical species where there are no constraints on the values. Note there is no factor of $\frac{1}{2}$ in these equations as the operator form is strictly (for example) $\pm \frac{1}{2} (g_{\text{JT}3}^x\hat{q}_{3x}\hat{q}_{4x} + g_{\text{JT}3}^y\hat{q}_{3y}\hat{q}_{4y}) - \hat{q}_{3y}\hat{q}_{4y} - \hat{q}_{3y}\hat{q}_{4y}$, which simplifies as above. Neither of the $g_4$ or $g_{\text{JT}4}$ terms are significant, but are included for completeness.

Comparing the parameters derived here with those of Allen et al. shows some differences, with the linear Jahn–Teller terms rather larger in the current analysis. The differences arise from the different models used; the earlier work included a $x_{44}$ anharmonicity term, but here this is forced to zero. The previous work included a linear vibrational (in $g_4$) term mixing in a third electronic state but this can be shown to be equivalent to a quadratic Jahn–Teller effect in $g_4$ using second order perturbation theory. We do not have enough experimental information to discriminate between the models, but they do confirm that, within a factor of two, sensible values are being used for the Jahn–Teller parameters.

### Predictions for other isotopologues
Given the force field and Jahn–Teller terms expressed in internal coordinates, it is now straightforward to repeat the process above for all the other isotopologues. The normal coordinate parameters for all four species are given in Tables 8 to 11.

Note that, as discussed above, in the less symmetrical species, a linear potential term is introduced into what is formally the symmetric stretch, $\nu_3$, but not the umbrella mode, $\nu_2$ as it has the wrong symmetry.

For the quadratic terms, several small terms have been included for completeness; the biggest addition in $C_{2v}$ symmetry is likely to come from the $g_{11}$ and $g_{11}$′ terms adding to $g_3$.

Given the normal mode parameters the vibronic energy levels can be calculated for all 4 isotopologues, and selected levels are listed in Table 12. A matrix diagonalization was required, as described above for NH$_3$, with the basis sets slightly adjusted to ensure convergence, mainly requiring a higher limit in $v_4$ for the $C_{2v}$ species. The notation used to label the vibrational states is such that $n^m$ implies $m$ quanta in mode $n$. For the degenerate modes the quanta along the $x$ and $y$ modes are given separately. The quantum numbers are based on the largest coefficients in the final wavefunctions; this normally gives a clear assignment, though for NH$_2$D a level at 3289.7 cm$^{-1}$ could also have been assigned to the $3\gamma^1 A_2$ level.

### Table 7
<table>
<thead>
<tr>
<th>$r_{\text{NH}}$/Å</th>
<th>1.040</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{\text{stretch}}$/(au Å$^{-2}$)</td>
<td>5.4344</td>
</tr>
<tr>
<td>$k_{\text{bend}}$/au</td>
<td>0.3820</td>
</tr>
<tr>
<td>$k_{\text{umb}}$/au</td>
<td>0.1871</td>
</tr>
<tr>
<td>$g_{\text{JT}3}$/(au $^{-1}$)</td>
<td>0.1659</td>
</tr>
<tr>
<td>$g_{\text{JT}4}$/(au $^{-1}$)</td>
<td>0.0339</td>
</tr>
<tr>
<td>$g_{\text{JT}3}$/(au $^{-2}$)</td>
<td>0.3982</td>
</tr>
</tbody>
</table>

### Table 8
| Calculated normal mode frequencies (cm$^{-1}$) for the $B^1E^*$ state of ammonia |
|------------------|--------|
| $\omega_1$ | 3025.2 |
| $\omega_2$ | 892.5 |
| $\omega_{3x}$ | 3188.0 |
| $\omega_{3y}$ | 1397.5 |
| $\omega_{4x}$ | 1166.4 |
| $\omega_{4y}$ | 1277.0 |
| NH$_3$ | 3025.2 |
| NH$_3$D | 3089.2 |
| ND$_3$H | 2208.5 |
| ND$_3$ | 2140.0 |

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To understand the pattern of levels produced by this model, it is helpful to consider the pattern of levels associated with just the stretching modes, and with terms involving bending omitted. Fig. 3 shows the stretching levels for NH₃ with the various possible combinations of Jahn–Teller terms. Note there are two stretching levels, one of which (the asymmetric stretch, \( v_3 = 1 \)) is notionally four fold degenerate, and this is split into two components by the linear Jahn–Teller effect and three by the quadratic Jahn–Teller effect. The other stretching level and the origin remain degenerate. Contrast this with the corresponding energy level diagram for ND₂H in Fig. 4, where there are now three stretching levels, each notionally doubly degenerate. The linear Jahn–Teller effect shifts all these levels, but does not lift the degeneracy; this has been discussed above as being a consequence of the form of the Hamiltonian, and is not specific to this case. The largest effect of the linear term is to increase the separation between the two N–H stretching levels, with smaller shifts of the other levels. The quadratic term does lift all the degeneracies, to a small amount for the origin level and rather more for the other levels. Fig. 5 shows the pattern of stretching levels for all four species, calculated including both linear and quadratic terms. The overall trend is of course to lower energies as modes switch from N–H to N–D stretches but the pattern associated with each set of bonds is similar, so the levels associated with the N–H bonds in NH₂D is replicated in the N–D bonds in ND₃H. The switch from N–H to N–D can also be seen in the detailed form of the normal coordinates – see Table S3 in the supplementary notes (ESI†).

The only information that we can directly compare with experiment is the origin band, though it would be very informative to have spectra of the excited stretching levels. The key
prediction is that the origin level should be split by a relatively small amount, with the sign of the splitting opposite for NH2D and ND3H. Our observations reflect this, with the splittings having the right relative sign (+34.8 and −35.1 cm−1) though about twice the magnitude of the calculated splittings (+18.6 and −18.5 cm−1). Given we have no information on the quadratic Jahn–Teller effect for the bending modes, and that the analysis above shows that the quadratic term is the key contributor to the splitting of the origin band, this is good agreement. Interestingly, the observation of equal and opposite splittings was seen in the two other studies that are directly comparable to this one by Yu et al.7 on the X1E1u state of C6D6D (−8.9557 cm−1) and C5D6H (+9.1975 cm−1) and by Melnik et al.8,9 on the X2E state of CH2DO (−45.66 cm−1) and CD3HO (+43.44 cm−1).

At this stage it is worth comparing the approach taken here with the approaches taken by previous workers in calculating this splitting. Perhaps the earliest consideration of this issue is in the interpretation of ESR data on the negative ions of benzene-d1 and cyclo-octatetraene-d1 by Carrington et al.17 They suggested that kinetic energy considerations would give different zero point energies for different structures. This approach would obviously be appropriate where the different structures had significant barriers between them, but it is less clear that it would be appropriate for cases like the current one where the barriers are much less than the zero point energy.

Later works undertook more detailed measurements and calculations. Scharf et al.18 considered various 1E1u Rydberg states of asymmetrically deuterated benzene. Unfortunately the experimental data available is low resolution, so the zero point splittings cannot be directly measured, but only inferred from different widths of vibronic bands. The experimental results showed a progression in only one of the four possible Jahn–Teller active modes, so only that mode was modelled in a way similar to that done here. The model was restricted to linear terms, and as the mode involved ring distortion $\alpha$ and $k$ for that mode were assumed unchanged on deuteration. The zero point splitting was accounted for by a single parameter, $E_s - E_p$, giving the difference in zero point energy in the other modes; in our notation this is equivalent to $\Delta g_{ss} \approx g_{pp}$ summed over all the other modes. The derivation above shows this is equivalent to assuming the zero point splitting arises from a quadratic Jahn–Teller effect in these other modes. They produced estimates of this quantity based on a simple orbital model. The difference from our model stems partly from choice of definitions, in that they do not consider this difference in second derivatives to be a Jahn–Teller effect.

Eiding and Domcke19 looked at partial isotopic substitution in the X2E1u and B2E3u states of 1,4-C6H4D2−. They modelled the energy levels including several Jahn–Teller active modes using an approach similar to that taken in this paper, but with one essential difference. The reference point taken for all the derivatives with respect to energy was the geometry of the neutral molecule, rather than the ion. (This approach was taken as the aim of that study was to calculate photoelectron spectra.) This meant that the first derivatives with respect to energy for symmetric modes was non zero even in the absence of the Jahn–Teller effect, giving terms comparable in magnitude to the linear Jahn–Teller terms. Values for these first derivatives for both the symmetrical and Jahn–Teller active modes were taken from derivatives of orbital energies from a Hartree–Fock calculation on the neutral molecule. Quadratic Jahn–Teller effects were not included, but the calculations did nevertheless predict non-zero zero-point splittings for the lower symmetry isomers. This does not contradict the result of Iwahara et al.10 that linear terms do not give a zero-point splitting due to the use of a non-equilibrium reference geometry as the transformation to the equilibrium geometry will introduce additional terms in the Hamiltonian that are effectively quadratic.

The study by Yu et al.7 on the X1E1u state of C6H6D and C5D6H used a similar approach to that used here, but only considered linear Jahn–Teller terms. In addition, in C6H6 the mode with the largest Jahn–Teller activity was thought to involve distortion of the carbon framework, so they assumed deuterium substitution did not alter the Jahn–Teller parameters and so gave no zero point splitting in that mode. They suggested, following Scharf et al.18 that there could be a zero point energy difference in the other modes, but did not attempt to calculate this, and the current work suggests there should not be any if only linear terms are included. They did not discuss quadratic Jahn–Teller effects in detail, though they pointed out that it was shown by Engelage and Lineberger20 using symmetry arguments that the quadratic Jahn–Teller effect must be zero in $D_{3h}$ symmetry. Iwahara et al.10 discuss this specific case, and attribute the splitting of the zero point level to quartic potential terms. Higher terms would give five equivalent minima, and more recent calculations of Bearpark et al.21 suggests that the barrier between them is only 3.6 cm−1. Bearpark et al.21 did calculate zero point energies for different equilibrium structures with differences of the right order of magnitude to the observations, but they do not indicate which value should be compared to the experiment. It is anyway not clear that the approach they used, considering a single electronic surface, will give correct results, especially as they ignored anharmonic effects which must be important with such a surface given the low barrier (3.6 cm−1) in one mode.

Melnik et al.8,9 looked at submillimeter spectra of the lowest vibrational level of the X2E state of CH2DO and CHD2O radicals. Their approach to the rotational energy level pattern was similar to that taken here, though made rather more complicated by the presence of a spin–orbit interaction. They describe a method of calculating the zero point splitting, based on calculating zero point energies at two different minima on the lowest potential energy surface with the unique atom on or off the symmetry plane. This gives promising results, predicting approximately equal and opposite splittings for CH2DO and CHD2O and of the correct order of magnitude, though rather sensitive to the $ab initio$ method used. They do however comment that their method relies on a harmonic frequency model and ignores the upper surface and the coupling between them, and acknowledge that they may be relying on a cancellation of errors, as again the mild Jahn–Teller interaction implies a very anharmonic mode.
It is also worth mentioning work on the ground state of CH₄⁺, for which significant work on partially deuterated isotopologues has been done²²,²³ but this is less directly comparable as the starting model is minima with a large barrier between them, though with significant tunnelling between them and thus requiring a different theoretical approach.

**Rotational constants**

Given a model based on normal co-ordinates, it is also possible to predict the rotational constants for the origin levels by taking into account the variation of the rotational constants with vibrational co-ordinate. The key quantity is the \( \mu_{ab} \) tensor, essentially the inverse of the moment of inertia tensor, with the diagonal terms the rotational constants. For a summary of the theory and notation see the review by Mills.²⁴ There are many possible terms, of which we discuss the three largest here. We start with the term involving the second derivative of this \( \mu_{ab} \) tensor:

\[
\sum_{s\neq r} \mu_{ab}^{(rs)} \dot{q}_r \dot{q}_s \dot{J}_x \dot{J}_y
\]  
(41)

Here \( \dot{a} \) and \( \dot{b} \) label the principal axes, \( r \) and \( s \) the normal modes and \( \mu_{ab}^{(rs)} \) is the second derivative with respect to normal co-ordinates \( r \) and \( s \):

\[
\mu_{ab}^{(rs)} = \frac{\partial^2 \mu_{ab}}{\partial q_r \partial q_s}
\]  
(42)

which can in turn be calculated from the first derivative of the moment of inertia tensor with respect to normal co-ordinates. For \( r = s \) this term has matrix elements diagonal in vibrational state proportional to \( v + \frac{1}{2} \) and is one of the conventional contributions to the vibrational dependence of the rotational constants.

The vibrational angular momentum also contributes to this vibrational dependence; the relevant term is:

\[
-2 \sum \beta \omega_\beta \bar{\xi}_\beta \dot{J}_\beta
\]  
(43)

where \( \bar{\xi}_\beta \) is the vibrational angular momentum:

\[
\bar{\xi}_\beta = \sum_{\gamma} \gamma (a \beta) \sqrt{c_{\gamma \beta}} \dot{q}_\gamma \dot{J}_\beta
\]  
(44)

This does not directly contribute to the rotational constants as the Coriolis coupling constant, \( \gamma_{\alpha \beta} \), is zero if \( r = s \), but second order perturbation theory does give a term containing \( \dot{J}_\alpha \dot{J}_\beta \) and thus a vibration state dependent contribution to the rotational constants, another conventional contribution to the vibrational constants.

The final conventional contribution involves the first derivative of the \( \mu_{ab} \) tensor:

\[
\sum_{s \neq r} \mu_{ab}^{(r)} \dot{q}_r \dot{J}_x \dot{J}_y
\]  
(45)

where

\[
\mu_{ab}^{(r)} = \frac{\partial \mu_{ab}}{\partial q_r}
\]  
(46)

As this contains a single vibrational operator it has selection rules \( \Delta v = \pm 1 \), and thus no diagonal contribution. In conventional theory it becomes significant when combined with cubic anharmonic potential terms, and thus gives a vibrational contribution in second order. In the current analysis we are ignoring anharmonicity, but the Jahn–Teller potential terms also have selection rules \( \Delta v = \pm 1 \), and so provide a similar contribution in second order.

To calculate all the above contributions all that is required are the Coriolis coupling constants and the derivative of the moment of inertia tensor with respect to the normal co-ordinates. Both of these are readily available from the normal co-ordinate analysis. Rather than attempting to derive general expressions for the variation of rotational constants with \( v \) using perturbation theory we choose to use a simpler numerical approach as follows:

For each rotational constant, for which the effective operator is \( \mu_{ab} \dot{J}_a \dot{J}_b \) we set up the matrix of \( \sum_{r \neq s} \mu_{ab}^{(rs)} \dot{q}_r \dot{q}_s \) (from (41)) and

\[
\sum \mu_{ab}^{(r)} \dot{q}_r \]  
(45) evaluated in the harmonic basis described above used to calculate the vibronic energy levels. Transforming this matrix using the eigenvectors from the energy level calculations then gives the corrections to the rotational constants as the diagonal elements. For the Coriolis terms, eqn (43) a slightly different approach is required. For each principal axis, \( a \), we set up the matrix of \( -2 \mu^{(3)}_a \bar{\xi}_a \dot{J}_b \) in the same harmonic basis, and again transform this using the eigenvectors from the energy level calculations. The corrections to a rotational constant, \( \Delta \mu_{ab} \) for a given level \( i \) are then calculated using second order perturbation theory:

\[
\Delta \mu_{ab} = \sum_{i,j} M_{ij}^{(3)} M_{ij}^{(3)} \frac{E_i - E_j}{E_i - E_j}
\]  
(47)

where \( M_{ij}^{(3)} \) is the transformed matrix and the energies, \( E_i \), are the exact vibronic energy levels as in Table 12. This formula can fail for near degenerate levels, so the program excludes states from the sum where \( |E_i - E_j| < 1 \text{ cm}^{-1} \) and \( M_{ij} \) is non-zero, and displays these separately. This test does not exclude any contributions in the current calculations.

The above is expressed in terms of a general tensor element \( \mu_{ab} \), though symmetry is sufficiently high for all the off-diagonal terms \( (\beta \neq \alpha) \) to be zero, though the \( ab \) term gives a term between the \( |+\rangle \) and \( |\rangle \) components. In the absence of the Jahn–Teller effect, the sum of the terms described above gives essentially the same result as the harmonic formula for the vibrational dependence of the rotational constants.²⁴ Table 13 gives the calculated and observed values for the rotational constants of the zero point levels.

Perhaps the most striking feature of the table is that even for the symmetrical isotopologues the \( A \) and \( B \) rotational constants are different, in contradiction to what is expected for a symmetric top. In fact the rotational Hamiltonian used already allows for this through the term in \( q \). This can be seen if the standard asymmetric top Hamiltonian is re-expressed in a form close to the symmetric top form:
Elements between the two components: $A$, $B$

We have used $J_{\alpha}^{\beta}$, $H_{\alpha}$, and so on and so forth.

Observed values are from this paper and ref. 5.

Table 13 Calculated rotational constants (cm$^{-1}$) for zero point levels of the $B^1E^*$ state of ammonia

<table>
<thead>
<tr>
<th>NH$_3$</th>
<th>NH$_2$D</th>
<th>ND$_2$H</th>
<th>ND$_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2$</td>
<td>$B_1$</td>
<td>$A_2$</td>
<td>$B_1$</td>
</tr>
<tr>
<td>C</td>
<td>5.180</td>
<td>5.180</td>
<td>3.969</td>
</tr>
<tr>
<td>$\frac{1}{2}(A+B)$</td>
<td>10.537</td>
<td>10.537</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}(A-B)$</td>
<td>0.317</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Observed values:

<table>
<thead>
<tr>
<th>NH$_3$</th>
<th>NH$_2$D</th>
<th>ND$_2$H</th>
<th>ND$_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>9.856</td>
<td>10.780</td>
<td>8.377</td>
</tr>
<tr>
<td>B</td>
<td>10.484</td>
<td>6.766</td>
<td>6.455</td>
</tr>
<tr>
<td>C</td>
<td>5.1565</td>
<td>3.953</td>
<td>3.167</td>
</tr>
<tr>
<td>$q$</td>
<td>0.8868</td>
<td>0.681</td>
<td>0.481</td>
</tr>
</tbody>
</table>

$\hat{H}_{\text{rot}} = \frac{\hbar}{2}J_a^2 + \frac{\hbar}{2}J_b^2 + C\hat{J}_c^2$

$= \frac{1}{2}(A+B)(J_a^2 + J_b^2) + \frac{1}{2}(A-B)(J_a^2 - J_b^2) + C\hat{J}_c^2$

Applying the inverse of transformation (10):

$\langle A = 1 | \hat{H}_{\text{rot}} | A = 1 \rangle = \frac{1}{\sqrt{2}}(|+\rangle + |\rangle)$; $\langle A = -1 | \hat{H}_{\text{rot}} | A = 1 \rangle = \frac{1}{\sqrt{2}}(|+\rangle - |\rangle)$

the terms in $\hat{H}_{\text{rot}}$ giving matrix elements diagonal in vibrionic state are simply the standard symmetric top terms, as $A-B$ for the $|\rangle$ component is equal to $B-A$ for the $|\rangle$ component and the term in $J_a^2 - J_b^2$ vanishes. However $\hat{H}_{\text{rot}}$ does give matrix elements between the two components:

$\langle A = -1 | \hat{H}_{\text{rot}} | A = 1 \rangle = \frac{1}{2}(|+\rangle \langle H_{\text{rot}} | +\rangle - \langle H_{\text{rot}} | -\rangle)$

$= \frac{1}{2}(A_2 - A_1)(J_a^2 - J_b^2)$

We have used $A_1$ for the $A$ rotational constant for the $|\rangle$ component, and so on and so forth.

the corresponding matrix element of the term in the symmetric top Hamiltonian:

$q(J_a^2L_a^2 + J_b^2L_b^2)$ (51)

allows the identification $q = \frac{1}{2}(A_1 - A_2)$. Note that this term will also give a term mixing the $|\rangle$ and $|\rangle$ components, but we do not work through this here.

Comparing the calculated and observed values for $q$ for NH$_3$ and ND$_3$ reveals a factor of more than 2 between them. There are two factors that make this disagreement unsurprising. One is that our knowledge of the Jahn–Teller terms is incomplete; in this case it is the linear Jahn–Teller term that dominates $q$. Secondly there can also be a significant electronic contribution to this term, and indeed the previous paper explained the term in $q$ entirely as a rotational-electronic effect. The numbers here suggest both contributions may be significant. This is confirmed by looking at the mixed isotopologues, where changes of the right order of magnitude in the rotational constants are predicted, but not with the right direction. There is no simple way of calculating the electronic effects, though the order of magnitude is expected to be similar to the vibrational terms in this case. The overall conclusion on the rotational structure is that small ($\sim 10\%$) differences between the rotational constants for the two components of the electronic state are to be expected, with significant contributions from interactions both within the electronic state and with other electronic states.

Conclusions

The model developed to analyse the rotational structure in the $B^1E^*$ state of NH$_2D$, and confirms the validity of the model, and that for cases of a mild Jahn–Teller effect the formal lifting of the degeneracy by partial isotopic substitution leaves significant terms in the Hamiltonian mixing the two components of the state that must be included for a satisfactory simulation of the rotational structure. The
observation of approximately equal but opposite splittings in the zero point level for complementary isotopic substitution (here NH$_3$D compared to NHD$_3$) seen before in the literature is further reinforced here. We present a method of calculating this from the Jahn–Teller parameters from any one isotope, though we are limited from quantitative agreement here by a lack of a full knowledge of the required Jahn–Teller parameters. This limitation also applies to our calculation of the rotational constants, which are affected by the Jahn–Teller effect.

Our analysis confirms the result of Iwahara et al.\textsuperscript{10} that the zero point splitting must come from quadratic or higher Jahn–Teller effects and a linear term is not sufficient to break the symmetry. This is not clear from previous literature analysis of such splittings, which have concentrated on Jahn–Teller active modes that would not be expected to change on isotopic substitution. More recent studies have also concentrated on properties derived from substitution. More recent studies have also concentrated on Jahn–Teller active symmetry. This is not clear from previous literature analysis of Jahn–Teller effects and a linear term is not sufficient to break the zero point splitting must come from quadratic or higher Jahn–Teller parameters. This limitation also applies to our calculation of the rotational constants, which are affected by the Jahn–Teller effect.

Acknowledgements

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