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Universality and Short-Wavelength Approximations for Chaotic Wave Scattering

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Abstract—We give an overview of wave scattering in open cavities in which the ray dynamics is chaotic. In the limit of short wavelengths certain properties emerge that are universal and do not depend on the details of the cavity. These universal features are described by random matrix theory. We discuss in particular results that characterize the transmission probabilities and transmission times of waves through the cavity. Short-wavelength approximations that use statistical properties of long rays are able to explain this universality.

I. OPEN CHAOTIC CAVITIES

Chaotic cavities are cavities in which the ray dynamics is chaotic. They find applications, for example, in microwave physics, in acoustics or in mesoscopics [1]–[5]. Here we give a brief overview of wave transport through chaotic cavities that are opened up by attaching semi-infinite leads. An example is shown in Fig. 1 in the form of a quarter stadium. These systems are motivated by the transport through quantum dots [5], [6]. We concentrate on the case of two leads, but the formalism can be generalised to an arbitrary number of leads.

Stationary waves inside the cavity and the leads satisfy the Helmholtz equation

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{r}) = 0,$$

(1)

where $k$ is the wavenumber. In addition one has to require appropriate boundary conditions, for example Dirichlet boundary conditions for which the wave function $\psi(\mathbf{r})$ vanishes at the boundary of cavity and leads.

One way to characterise the wave solutions for this system is to consider the corresponding scattering problem. In each lead there is a finite number of incoming and outgoing modes. These numbers are given by $M_i = \left[kw_i/\pi\right]$ where $w_i$ is the width of the $i$-th lead.

The $M \times M$ scattering matrix $S$ connects the $M$ (flux normalised) incoming modes to the $M$ outgoing modes, where $M = M_1 + M_2$. Due to flux conservation $S$ is unitary, $S^\dagger S = 1$, and it has the block structure

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix},$$

(2)

Here $r$ and $t$ refer to reflection and transmission for incoming waves in lead 1, and $r'$ and $t'$ refer to reflection and transmission for incoming waves in lead 2. There are a number of quantities that can be obtained from the $S$-matrix and that can be used to characterise the transmission through the cavity. Some of them are discussed in the following section.

II. TRANSMISSION EIGENVALUES AND TIME DELAYS

The total transmission coefficient $T$ for incoming waves in lead 1 is

$$T = \sum_{a=1}^{M_1} \sum_{b=1}^{M_2} |s_{ba}|^2 = \text{Tr}(tt^\dagger).$$

(3)

Similarly, the reflection coefficient is $R = \text{Tr}(rr^\dagger)$, and unitarity of $S$ requires that $R + T = M_1$.

The eigenvalues of $tt^\dagger$ are the transmission eigenvalues

$$T_1, \ldots, T_n, \quad T_j \in [0, 1], \quad n = \min(M_1, M_2)$$

Many important transmission properties can be obtained from the eigenvalues $T_j$. For example, the transmission coefficient $T = \sum_j T_j$, the variance of $T$, the shot noise $(\sum_j T_j (1-T_j))$ or the transmission moments $\frac{1}{n} (\sum_j T_j^n)$. The expression of transmission properties in terms of the transmission eigenvalues is often referred to as Landauer-Büttiker formalism [7], [8].

A second set of quantities are related to the Wigner-Smith matrix $Q$ [9], [10]

$$Q = -i S^\dagger \frac{\partial S}{\partial E}.$$ 

(4)

The matrix $Q$ is hermitian, $Q = Q^\dagger$, and its $M$ eigenvalues are the proper time delays $\tau_j$. They characterise temporal aspects of a wave scattering process. Similar as with the transmission eigenvalues $T_j$, the proper time delays $\tau_j$ can be used to express other quantities of interest. Among them are the Wigner time delay $\tau_W = \frac{1}{M} \text{Tr} Q = \frac{1}{M} \sum_j \tau_j$.
the variance of $\tau_W$ and the moments of the proper time delays $\frac{1}{M} \langle \sum_j \tau_j^k \rangle$. Interesting properties of the Wigner time delay include its relation to the total scattering phase shift $\tau_W(E) = -\frac{1}{M} \frac{d}{dE} \ln \det S(E)$ and to the density of states $\tau_W(E) = \frac{2\pi}{\hbar} d(E)$.

A remarkable property of chaotic cavities is that the statistical distributions of the transmission eigenvalues $T_j$ and the proper time delays $\tau_j$ are expected to become universal in the limit of short wavelengths ($k \to \infty$) if the leads are sufficiently thin. These universal distributions are described by random matrix theory. This is discussed in the following.

### III. Random Matrix Theory

There are two different approaches for applying random matrix theory to open chaotic cavities. In one approach the $M \times M$ scattering matrix is related to an $N \times N$ Hermitian matrix $H$ that describes the eigenmodes of the corresponding closed cavity (without leads), and an $N \times M$ coupling matrix $V$ that describes the coupling between inside and outside [11]. The relation is given by

$$S(E) = I - iV \frac{1}{E - H_{\text{eff}}} V^\dagger, \quad H_{\text{eff}} = H - \frac{i}{2}V V^\dagger. \quad (5)$$

The matrix $H$ can be chosen, for example, as a random Gaussian matrix and the matrix $V$ can either be taken fixed or random [12].

In the second approach the scattering matrix is modelled directly by a random matrix. Derived from an information theoretic approach, the corresponding distribution of the $S$-matrix is uniquely parametrised by the average scattering matrix $\bar{S}$ and is described by the so-called Poisson kernel [13]. For perfect coupling ($\bar{S} = 0$) the relevant ensembles are the circular ensembles.

The two approaches for the scattering matrix can be shown to be equivalent in some cases [14].

For the transmission eigenvalues the results are the following. The joint probability density function of the $\gamma_j$ is given by the Laguerre ensemble which has the form [17]

$$P(\gamma_1, \ldots, \gamma_n) = N_\beta \prod_{j=1}^n \gamma_j^{\beta M/2} e^{-\beta \gamma_j/2} \prod_{1 \leq j < k \leq n} |\gamma_j - \gamma_k|^\beta. \quad (8)$$

The applicability of random matrix theory to chaotic cavities was first established empirically, but it was later confirmed by asymptotic methods in the limit of short wavelengths. The following section discusses these asymptotic approaches.

### IV. Approximations in the Limit of Short Wavelengths

The elements $t_{ba}$ of the transmission matrix $t$ can be approximated in the short-wavelength limit $k \to \infty$ in terms of rays that go from the incoming lead to the outgoing lead [18]$

$$t_{ba} \approx \sum_{\gamma:a \to b} A_\gamma \exp(i k L_\gamma). \quad (9)$$

An example of such a ray is shown in Fig. 1. The incoming and outgoing channels $a$ and $b$ determine the modulus of the angles with which the rays enter and leave the cavity (with respect to the normals at the openings)

$$\sin \theta_1 = \pm \frac{a \pi}{k w_1}, \quad \sin \theta_2 = \pm \frac{b \pi}{k w_2}, \quad (10)$$

where $a \in \{1, \ldots, M_1\}$ and $b \in \{1, \ldots, M_2\}$. The sum over $\gamma$ in (9) runs over the infinite number of such trajectories, $L_\gamma$ is the length of the ray $\gamma$ and $A_\gamma$ an amplitude factor that depends on stability properties.

We discuss in the following how this approximation can be applied in order to reproduce the random matrix result (7) for the transmission coefficient $T$. The approximation for $T$ is given by

$$T = \left( \sum_{a,b} t_{ba} t_{ba}^{\dagger} \right)_k \approx \sum_{a,b,\gamma: a \to b} A_\gamma \bar{A}_\gamma^{\dagger} \exp(i k (L_\gamma - L_{\gamma'}))_k. \quad (11)$$

This expression contains a double sum over rays $\gamma$ and $\gamma'$. It involves a local average over the wavenumber $k$ to smooth out fluctuations and to obtain the mean transmission. A central observation is that most terms in this double sum do not contribute because the average over the wavenumber $k$ removes most of these highly oscillatory terms. The only important contributions come from pairs of rays that are correlated.

In a first approximation expression (11) was considered in the diagonal approximation where $\gamma = \gamma'$ [19]

$$T_{\text{diag}} = \sum_{a,b} \sum_{\gamma: a \to b} |A_\gamma|^2. \quad (12)$$

This approximation can be evaluated by using a classical sum rule that is based on average properties of long rays in an open chaotic cavity. It can be obtained by considering the probability density of rays that enter the cavity with angle
lead 1

summing over various diagrams that expressed all possible short-wavelength limit to a purely combinatorial problem of rules. The diagrammatic rules reduced the calculations in the $T$ of rays to evaluation in \([22]\), \([23]\) was the realisation that contributions complete expansion of the transmission coefficient $T$ and arbitrary links between them, and they reproduce the involvement pairs of rays with arbitrary many self-encounters also generalising methods for closed systems \([24]\), \([25]\). They give the next-to-leading order term in systems \([21]\), and they give the next-to-leading order term in opposite directions. The contributions of these pairs of rays were evaluated in \([20]\), generalising methods for closed systems \([22]\), \([23]\).

A schematic view of such a correlated pair of rays is shown in Fig. 2. The rules are the following: there is a factor $(-M)$ for every encounter, and a further factor of $M_1M_2$ for the number of incoming and outgoing leads. The pair of rays in Fig. 2 have three links and one encounter and their contribution is

$$
\left( \frac{1}{M} \right)^3 (-M) M_1M_2 = -\frac{M_1M_2}{M^2}.
$$

This is in deed the second term in the expansion (7). Note that these diagrammatic rules were obtained from the properties of long rays in chaotic systems, similarly as the classical sum rule in (13).

In summary, the following results for the transmission moments $\mathcal{M}_k = \frac{1}{2} \langle \sum_j T_j^k \rangle$ were obtained: The first correction to the first moment $\mathcal{M}_1$ \([20]\), the leading order for the second moment $\mathcal{M}_2$ \([26]\), all orders for the first and second moment $\mathcal{M}_1$ and $\mathcal{M}_2$ \([22]\), \([23]\), the leading order of all moments $\mathcal{M}_k$ \([27]\), the second order of all moments $\mathcal{M}_k$ \([28]\), and finally all orders of all moments $\mathcal{M}_k$ \([29]\)--\([32]\). All results are in agreement with random matrix theory.

For the proper time delays the results are not as complete, as is discussed in the following. One can base short-wavelength approximations for the moments of the proper time delays on the definition of the Wigner-Smith matrix in (4). This involves the same type of lead-connecting rays as for the transmission moments. However, one does not have similar diagrammatic rules in this case and this makes the calculations more complicated. Results that have been obtained for the moments of the proper time delays $m_k = \frac{1}{N} \langle \sum_j \tau_j^k \rangle$ in this way are: All orders for the first moment $m_1$ \([33]\), the leading order for all moments $m_k$ \([34]\), and the first two orders for all moments $m_k$ \([28]\).

Recently, however, a different approximation in the short-wavelength limit has been derived for the time delays \([35]\). It is based on the so-called resonance approximation for the Wigner-Smith matrix \([36]\)

$$Q = \hbar VT \uparrow \left( \frac{1}{(E - \mathcal{H}_{\text{eff}})^\dagger} \right) \frac{1}{(E - \mathcal{H}_{\text{eff}})} V,$$

and it involves rays that start in a lead and end inside the cavity. With this new approximation it has now been possible to establish diagrammatic rules for the evaluation of ray correlations, and they have been applied to derive all orders for the second moment $m_2$ and the leading five orders for all moments $m_k$ \([35]\). A remaining open problem is to obtain all orders for all moments $m_k$ as was possible for the transmission eigenvalues.

References
