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A note on balanced independent sets in the cube

Ben Barber

Department of Pure Mathematics and Mathematical Statistics
Centre for Mathematical Sciences
Wilberforce Road, Cambridge, CB3 0WB
U.K.
b.a.barber@dpmms.cam.ac.uk

Abstract

Ramras conjectured that the maximum size of an independent set in the discrete cube $Q_n$ containing equal numbers of sets of even and odd size is $2^{n-1} - \binom{n-1}{(n-1)/2}$ when $n$ is odd. We prove this conjecture, and find the analogous bound when $n$ is even. The result follows from an isoperimetric inequality in the cube.

The discrete hypercube $Q_n$ is the graph with vertices the subsets of $[n] = \{1, \ldots, n\}$ and edges between sets whose symmetric difference contains a single element. The cube $Q_n$ is bipartite, with classes $X_0$ and $X_1$ consisting of the sets of even and odd size respectively. The maximum-sized independent sets in $Q_n$ are precisely $X_0$ and $X_1$. Ramras [3] asked: how large an independent set can we find with half its elements in $X_0$ and half in $X_1$? Call such an independent set balanced. The following result verifies the conjecture made by Ramras for the case where $n$ is odd.

Theorem 1. The largest balanced independent set in $Q_n$ has size

\[
2^{n-1} - 2\left(\frac{n-2}{(n-2)/2}\right) \quad \text{if } n \text{ is even},
\]

\[
2^{n-1} - \left(\frac{n-1}{(n-1)/2}\right) \quad \text{if } n \text{ is odd}.
\]

For a set $A$ of vertices of $Q_n$, write $N(A)$ for the set of vertices adjacent to an element of $A$. The maximal independent sets in $Q_n$ all have the form $A \cup (X_1 \setminus N(A))$ for some $A \subseteq X_0$. So for a maximum-sized balanced independent set we seek the largest $A \subseteq X_0$ for which

$|A| \leq |X_1 \setminus N(A)|$. 
We use the following isoperimetric theorem for even-sized sets, due independently to Bezrukov [1] and Körner and Wei [2] (see also Tiersma [4]). Recall that $x < y$ in the simplicial order on $Q_n$ if either $|x| < |y|$, or $|x| = |y|$ and $x < y$ lexicographically.

**Theorem 2** ([1], [2]). Let $A \subseteq X_0$, and let $B$ be the initial segment of the simplicial order restricted to $X_0$ with $|B| = |A|$. Then $|N(B)| \leq |N(A)|$, and $X_1 \setminus B$ is a terminal segment of the simplicial order restricted to $X_1$.

**Proof of Theorem 1.** We will exhibit an initial segment $A$ of the simplicial order restricted to $X_0$, and a terminal segment $B$ of the simplicial order restricted to $X_1$, with $N(A) \cap B = \emptyset$ and $|A| = |B|$ as large as possible. It follows from Theorem 2 that $A \cup B$ will be a maximum-sized balanced independent set.

The form of $A$ and $B$ depends on the residue of $n \mod 4$. For $n = 4k$ we take

\[
A = [n]^{(0)} \cup [n]^{(2)} \cup \ldots \cup [n]^{(2k-2)} \cup (12 + [3, n]^{(2k-2)})
\]

\[
B = (1 + [3, n]^{(2k)}) \cup [2, n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \ldots \cup [n]^{(n-3)} \cup [n]^{(n-1)},
\]

where, for instance,

\[
12 + [3, n]^{(2k-2)} = \{\{1, 2\} \cup x : x \subseteq \{3, 4, \ldots, n\}, |x| = 2k-2\}.
\]

For $n = 4k + 1$ we take

\[
A = [n]^{(0)} \cup [n]^{(2)} \cup \ldots \cup [n]^{(2k-2)} \cup (1 + [2, n]^{(2k-1)})
\]

\[
B = [2, n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \ldots \cup [n]^{(n-2)} \cup [n]^{(n)}.
\]

For $n = 4k + 2$ we take

\[
A = [n]^{(0)} \cup [n]^{(2)} \cup \ldots \cup [n]^{(2k-2)} \cup (1 + [2, n]^{(2k-1)}) \cup (2 + [3, n]^{(2k-1)})
\]

\[
B = [3, n]^{(2k+1)} \cup [n]^{(2k+3)} \cup \ldots \cup [n]^{(n-3)} \cup [n]^{(n-1)}.
\]

Finally, for $n = 4k + 3$ we take

\[
A = [n]^{(0)} \cup [n]^{(2)} \cup \ldots \cup [n]^{(2k)}
\]

\[
B = [n]^{(2k+3)} \cup \ldots \cup [n]^{(n-2)} \cup [n]^{(n)}.
\]

Verifying that these sets have the claimed sizes, and that $|A| = |B|$ in each case, is a simple application of the identities $\binom{m}{r} = \binom{m-1}{r-1} + \binom{m-1}{r}$, $\binom{m}{r} = \binom{m}{m-r}$ and $\sum_{r=0}^{m} \binom{n}{r} = 2^m$. \qed

The maximum-sized balanced independent sets constructed above are also maximal independent sets. For example, if $n = 4k + 3$, then any set not in the family is adjacent to a complete layer; the other cases are similar, with slight complications in the middle layers of the cube.
References


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