STABLE CATEGORIES AND RECONSTRUCTION

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Dedicated to the memory of Sandy Green

1. Introduction

The Green correspondence is a fundamental construction in modular representation theory of finite groups. It is expected (Broué’s abelian defect group conjecture for example) to be the shadow of a more structural categorical correspondence, yet to be found. In an inductive approach to this, a key case is when the Green correspondence induces a stable equivalence between blocks. This work is an attempt towards a Morita theory for stable equivalences between self-injective algebras. More precisely, given two self-injective algebras $A$ and $B$ and an equivalence between their stable categories, consider the set $S$ of images of simple $B$-modules inside the stable category of $A$. That set satisfies some obvious properties of Hom-spaces and it generates the stable category of $A$. Keep now only $S$ and $A$. Can $B$ be reconstructed? We show how to reconstruct the graded algebra associated to the radical filtration of (an algebra Morita equivalent to) $B$. It would be interesting to develop further an obstruction theory for the existence of an algebra $B$ with that given filtration, starting only with $S$ (this might be studied in terms of localization of $A_{\infty}$-algebras). Note that a result of Linckelmann [Li] shows that, if we consider only stable equivalence of Morita type, then $B$ is characterized by $S$ — but this result does not provide a reconstruction of $B$ from $S$.

We also study a similar problem in the more general setting of a triangulated category $\mathcal{T}$. Given a finite set $S$ of objects satisfying Hom-properties analogous to those satisfied by the set of simple modules in the derived category of a ring and assuming that the set generates $\mathcal{T}$, we construct a $t$-structure on $\mathcal{T}$. In the case $\mathcal{T} = D^b(A)$ and $A$ is a symmetric algebra, the first author has shown [Ri] that there is a symmetric algebra $B$ with an equivalence $D^b(B) \sim D^b(A)$ sending the set of simple $B$-modules to $S$. The case of a self-injective algebra leads to a slightly more general situation: there is a finite dimensional differential graded algebra $B$ with $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$ with the same property as above.

2. Notations

Let $\mathcal{C}$ be an additive category. Given $S$ a set of objects of $\mathcal{C}$, we denote by $\text{add} \, S$ the full subcategory of $\mathcal{C}$ of objects isomorphic to finite direct sums of objects of $S$.

Let $k$ be a field and $A$ a finite dimensional $k$-algebra. We say that $A$ is split if the endomorphism ring of every simple $A$-module is $k$. We denote by $A$-mod the category of finitely generated left $A$-modules and by $D^b(A)$ its derived category. For $A$ self-injective, we denote by $A$-stab the stable category, the quotient of $A$-mod by projective modules. Given $M$ an $A$-module, we denote by $\Omega M$ the kernel of a projective cover of $M$ and by $\Omega^{-1} M$ the cokernel of an injective hull of $M$. 
3. Simple generators for triangulated categories

3.1. **Category of filtered objects.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a full subcategory of $\mathcal{T}$.

We define a category $\mathcal{F}$ as follows.

- Its objects are diagrams
  
  $$\mathcal{M} = (\cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{\varepsilon_0} N_0)$$

  where $M_i$ is an object of $\mathcal{T}$, $M_i = 0$ for $i \gg 0$, such that

  (i) $M_1 \xrightarrow{f_1} M_0 \xrightarrow{\varepsilon_0} N_0$ is the beginning of a distinguished triangle

  (ii) for all $i \geq 1$, the cone $N_{i-1}$ of $f_i$ is in add $\mathcal{S}$

  (iii) the canonical map $\text{Hom}(N_0, S) \to \text{Hom}(M_0, S)$ is surjective for all $S \in \mathcal{S}$

  (iv) the canonical map $\text{Hom}(N_i, S) \to \text{Hom}(M_0, S)$ is bijective for all $S \in \mathcal{S}$ and $i \geq 1$.

  Note that $\varepsilon_i : M_i \to N_i = \text{cone}(f_{i+1})$ is well defined up to unique isomorphism for $i \geq 1$ thanks to property (iv). For $i \geq 0$, we define a new object $M_{\geq i}$ of $\mathcal{F}$ as $\cdots \to M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{\varepsilon_i} N_i$.

- Consider now $g_0 \in \text{Hom}_\mathcal{F}(M, M')$. By (iv), there are maps $h_0, h_1, \ldots$ and $g_1, g_2, \ldots$ making the following diagrams commutative

  $$\begin{array}{ccc}
  N_i[-1] & \xrightarrow{\rho_i} & M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{\varepsilon_i} N_i \\
  g_i[-1] & \xrightarrow{h_{i+1}} & M'_i \xrightarrow{\varepsilon'_i} N'_i
  \end{array}$$

  Here, $\rho_i : N_i[-1] \to M_{i+1}$ and $\rho'_i : N'_i[-1] \to M'_{i+1}$ are the maps making the horizontal rows in the diagram above into distinguished triangles.

  **Lemma 3.1.** The maps $g_i : N_i \to N'_i$ (for $i \geq 1$) depend only on $g_0$.

  **Proof.** We proceed by induction on $i$. We assume $g_{i-1}$ has been shown to depend only on $g_0$. Let us consider the lack of unicity of $h_i$. Consider $h_i, \tilde{h}_i : M_i \to M'_i$ such that $h_i \rho_{i-1} = \rho'_i g_{i-1}[1] = \tilde{h}_i \rho_{i-1}$. There is $p : M_{i-1} \to M'_i$ such that $h_i - \tilde{h}_i = p f_i$.

  By (iii) and (iv), there exists $q : N_{i-1} \to N_i$ such that $q \varepsilon_{i-1} = \varepsilon'_i p$. We have $\varepsilon'_i p f_i = q \varepsilon_{i-1} f_i = 0$, hence $\varepsilon'_i \tilde{h}_i = \varepsilon'_i h_i$.

  By (iv), we deduce that there is a unique map $g_i : N_i \to N'_i$ such that $g_i \varepsilon_i = \varepsilon'_i h_i$ and that map $g_i$ is the unique one such that $g_i \varepsilon_i = \varepsilon'_i h_i$. \hfill $\Box$

Let $g_0 \in \text{Hom}_\mathcal{F}(M, M')$, and $g'_0 \in \text{Hom}_\mathcal{F}(M', M'')$. We define the product $g_0 g'_0$ as the composition $N_0 \xrightarrow{g'} N'_1 \xrightarrow{g_0'} N''_{i+j}$.

  **Lemma 3.2.** Assume $\text{Hom}(S, T[n]) = 0$ for all $S, T \in \mathcal{S}$ and $n < 0$. Let $M$ be an object of $\mathcal{F}$. Then, the canonical map $\text{Hom}(N_0, S) \to \text{Hom}(M_0, S)$ is an isomorphism.
Proposition 3.4. \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0}) \) is a bounded t-structure on \( \mathcal{T} \).

Proof. By induction, we see there is no non-zero map from an object of \( \mathcal{T}^{\leq 0} \) to an object of \( \mathcal{T}^{> 0} \). Furthermore, we have \( \mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0} \) and \( \mathcal{T}^{> 0} \subseteq \mathcal{T}^{> 0}[1] \).
Let $N \in \mathcal{T}$. Pick a sequence as in Lemma 3.3. Take $s$ such that $d(s) > 0$ and $d(s + 1) \leq 0$. Let $L$ be the cone of $f_1 \cdots f_s : M_s \to N$. We have a distinguished triangle
\[ M_s \to N \to L \rightsquigarrow \]
with $M_s \in \mathcal{T}^{\leq 0}$ and $L \in \mathcal{T}^{> 0}$.

We have a characterization of $\mathcal{T}^{\geq 0}$ and $\mathcal{T}^{\leq 0}$:

**Proposition 3.5.** Let $N \in \mathcal{T}$. Then, $N \in \mathcal{T}^{\leq 0}$ if and only if $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$.

Similarly, $N \in \mathcal{T}^{\geq 0}$ if and only if $\text{Hom}(S[i], N) = 0$ for $S \in \mathcal{S}$ and $i > 0$.

**Proof.** We have $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$, if $N \in \mathcal{S}[r]$ with $r \geq 0$. By induction, it follows that if $N \in \mathcal{T}^{\leq 0}$, then $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$.

Assume now $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$. Pick a filtration of $N$ as in Lemma 3.3. Then, $d(1) \leq 0$, hence $d(i) \leq 0$ for all $i$ and $N \in \mathcal{T}^{\leq 0}$.

The other case is similar.

Note that the heart $\mathcal{A}$ of the $t$-structure is artinian and noetherian. Its set of simple objects is $\mathcal{S}$.

**Remark 3.6.** Assume $\mathcal{T}$ can be generated by a finite set of objects. Then, there is a finite subcategory $\mathcal{S}'$ of $\mathcal{S}$ generating $\mathcal{T}$. It follows immediately from condition (i) that $\mathcal{S} = \mathcal{S}'$. So, $\mathcal{S}$ has only finitely many objects.

3.2.2. In §3.2.2, we assume $\mathcal{T} = D^b(A)$ where $A$ is a finite dimensional $k$-algebra. By Remark 3.6, $\mathcal{S}$ is finite (note that $\mathcal{T}$ is generated by the simple $A$-modules, up to isomorphism).

**Proposition 3.7.** Let $S \in \mathcal{S}$. There is a bounded complex of finitely generated injective $A$-modules $I_S(S) \in \mathcal{T}^{\geq 0}$ such that, given $T \in \mathcal{S}$ and $i \in \mathbb{Z}$, we have
\[ \text{Hom}_{D^b(A)}(T, I_S(S)[i]) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases} \]

Similarly, there is a bounded complex of finitely generated projective $A$-modules $P_S(S) \in \mathcal{T}^{\leq 0}$ such that, given $T \in \mathcal{S}$ and $i \in \mathbb{Z}$, we have
\[ \text{Hom}_{D^b(A)}(P_S(S)[i], T) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** The construction of a complex $I_S(S)$ of $A$-modules with the Hom property is [Ri, §5] (note that the proof of [Ri, Lemma 5.4] is valid for non-symmetric algebras). It is in $\mathcal{T}^{\geq 0}$ by Proposition 3.5. Since $\bigoplus_{i \in \mathbb{Z}} \dim \text{Hom}_{D^b(A)}(V, I_S(S)[i]) = 0$ for all simple $A$-modules $V$, we deduce that $I_S(S)$ is isomorphic to a bounded complex of finitely generated injective $A$-modules.

The second case follows from the first one by passing to $A^{\text{opp}}$ and taking the $k$-duals of elements of $\mathcal{S}$.

We denote by $\tau^{> 0}$, etc... the truncation functors and $^!H^0$ the $H^0$-functor associated to the $t$-structure constructed in §3.2.1.

**Lemma 3.8.** The object $^!H^0(I_S(S))$ of $\mathcal{A}$ is an injective hull of $S$ and $^!H^0(P_S(S))$ is a projective cover of $S$.
Proof. We have a distinguished triangle

\[ \tau^0(I_S(S)) \to I_S(S) \to \tau^{-1}I_S(S) \to . \]

Let \( N \in \mathcal{A} \). We have \( \text{Hom}(N, \tau^{-1}I_S(S)) = 0 \) and \( \text{Hom}(N, I_S(S)[1]) = 0 \), so we deduce that \( \text{Hom}(N, \tau^0(I_S(S))[1]) = 0 \). It follows that \( \text{Ext}^1_{\mathcal{A}}(N, \tau^0(I_S(S))) = 0 \), hence \( \tau^0(I_S(S)) \) is injective. Since \( \text{Hom}(T, (\tau^{-1}I_S(S))[-1]) = 0 \), we have \( \text{Hom}(T, \tau^0(I_S(S))) \sim \text{Hom}(T, I_S(S)) = k^{\text{st}} \) for \( T \in \mathcal{S} \). So \( \tau^0(I_S(S)) \) is an injective hull of \( S \). The projective case is similar. \( \square \)

Let us consider the finite dimensional differential graded algebra

\[ B = \text{End}_A^*(\bigoplus_S P_S(S)) = \bigoplus_S \text{Hom}_A(\bigoplus_P P_S(S), \bigoplus_P P_S(S)[i]). \]

Denote by \( D^b(B) \) the derived category of finite dimensional differential graded \( B \)-modules.

**Theorem 3.9.** We have \( H^i(B) = 0 \) for \( i > 0 \) and \( i \ll 0 \). We have \( H^0(B)\text{-mod} \simeq \mathcal{A} \) and \( D^b(B) \simeq D^b(A) \).

**Proof.** Let \( N \in \mathcal{T} \) and consider a filtration of \( N \) as in Lemma 3.3. Take \( S \in \mathcal{S} \) such that \( S[i] \) is isomorphic to the cone of \( M_0 \to M_{i-1} \). Then, \( \text{Hom}(P_S(S)[i], N) \neq 0 \). It follows that the right orthogonal category of \( \{P_S(S)[i]\}_{S \in \mathcal{S}, i \in \mathbb{Z}} \) is zero. Since the \( P_S(S) \) are perfect, it follows that \( \bigoplus_S P_S(S) \) generates the category of perfect complexes of \( A \)-modules as a triangulated category closed under taking direct summands [Nee, Lemma 2.2]. The functor \( \text{Hom}_A^*(\bigoplus_S P_S(S), -) \) gives an equivalence \( D^b(A) \sim D^b(B) \) [Ke, Theorem 4.3].

Let \( C = \bigoplus_{S \in \mathcal{S}} P_S(S) \) and \( N = \tau^0(C) \). We have a distinguished triangle \( \tau^{-1}C \to C \to N \to . \) We have \( \text{Hom}(\tau^{-1}C, N[i]) = 0 \) for \( i \leq 0 \). We deduce that the canonical morphism \( \text{Hom}(N, N) \to \text{Hom}(C, N) \) is an isomorphism. We have \( \text{Hom}(C, (\tau^{-1}C)[i]) = 0 \) for \( i \geq 0 \) since \( \tau^{-1}C \) is filtered by objects in \( \mathcal{S}[d], d > 0 \) (cf Proposition 3.7). It follows that the canonical morphism \( \text{Hom}(C, C) \to \text{Hom}(C, N) \) is an isomorphism.

This shows that the canonical morphism \( \text{End}(C) \to \text{End}(\tau^0(C)) \) is an isomorphism. By Lemma 3.8, \( \tau^0(C) \) is a progenerator for \( \mathcal{A} \). So \( H^0(B)\text{-mod} \simeq \mathcal{A} \).

Note that \( H^i(B) = 0 \) for \( i \ll 0 \) because \( \bigoplus_S P_S(S) \) is bounded. Since \( P_S(S) \) is filtered by objects in \( \mathcal{S}[d] \) with \( d \geq 0 \), it follows from Proposition 3.7 that \( \text{Hom}(P_S(T), P_S(S)[i]) = 0 \) for \( i > 0 \). So, \( H^i(B) = 0 \) for \( i > 0 \). \( \square \)

The following proposition is clear.

**Proposition 3.10.** Let \( B \) be a dg-algebra with \( H^i(B) = 0 \) for \( i > 0 \) and for \( i \ll 0 \). Let \( C \) be the sub-dg-algebra of \( B \) given by \( C^i = B^i \) for \( i < 0 \), \( C^0 = \ker d^0 \) and \( C^i = 0 \) for \( i > 0 \). Then the restriction \( D(B) \to D(C) \) is an equivalence.

Let \( \mathcal{S} \) be a complete set of representatives of isomorphism classes of simple \( H^0(B)\text{-mod} \) (viewed as dg-C-modules). Then \( \mathcal{S} \) satisfies Hypothesis 1. Furthermore, \( \mathcal{A} \simeq H^0(B)\text{-mod} \).

So we have a bijection between

- the sets \( \mathcal{S} \) (up to isomorphism) satisfying Hypothesis 1
- the equivalences \( D^b(B) \to D^b(A) \) where \( B \) is a dg-algebra with \( H^i(B) = 0 \) for \( i > 0 \) and for \( i \ll 0 \) and where \( B \) is well-defined up to quasi-isomorphism and the equivalence is taken modulo self-equivalences of \( D^b(B) \) that fix the isomorphism classes of simple \( H^0(B)\text{-modules} \).
We recover a result of Al-Nofayee \[\text{[Al, Theorem 4]}\] :

**Proposition 3.11.** Assume \(A\) is self-injective with Nakayama functor \(\nu\). The following are equivalent:

- \(H^i(B) = 0\) for \(i \neq 0\)
- \(\nu(S) = S\) (up to isomorphism).

**Proof.** Note that \(S\) is stable under \(\nu\) if and only if \(\{P_S(S)\}_{S \in S}\) is stable under \(\nu\) (up to isomorphism). Given \(S, T \in S\) and \(i \in \mathbb{Z}\), we have

\[
\text{Hom}_{D^b(A)}(P_S(S), P_T(T)[i])^* \simeq \text{Hom}_{D^b(A)}(P_S(T), \nu(P_S(S))[-i]).
\]

If \(S\) is stable under \(\nu\), then \(\text{Hom}_{D^b(A)}(P_S(T), \nu(P_S(S))[-i]) = 0\) for \(i > 0\), hence \(H^{<0}(B) = 0\).

Assume now \(\nu(S)\) is stable under \(\nu\). Since it is perfect, it is isomorphic to a projective indecomposable module, hence to \(P_{S'}(S')\) for some \(S' \in S\). So, \(S\) is stable under \(\nu\).

We recover now the main result of [AlRi]:

**Corollary 3.12.** Let \(A\) be a self-injective algebra and \(B\) an algebra derived equivalent to \(A\). Then \(B\) is self-injective.

From Proposition 3.11, we recover [Ri, Theorem 5.1]:

**Theorem 3.13.** If \(A\) is symmetric then \(H^i(B) = 0\) for \(i \neq 0\), i.e., there is an equivalence \(D^b(A) \sim D^b(A)\) where \(S\) is the set of images of the simple objects of \(A\).

**Remark 3.14.** Theorem 3.13 does not hold in general for a self-injective algebra. Take \(A = k[\varepsilon]/(\varepsilon^2) \rtimes \mu_2\), where \(\mu_2 = \{\pm 1\}\) acts on \(k[\varepsilon]/(\varepsilon^2)\) by multiplication on \(\varepsilon\). Assume \(k\) does not have characteristic 2. This is a self-injective algebra which is not symmetric. The Nakayama functor sways the two simple \(A\)-modules \(U\) and \(V\).

Let \(P_U\) (resp. \(P_V\)) be a projective cover of \(U\) (resp. \(V\)). Take \(S = U\) and \(T = P_U[1]\). Then, the set \(S = \{S, T\}\) satisfies Hypothesis 1. We have \(I_S(T) \simeq T\) and \(I_S(S) \simeq 0 \to P_U \to P_V \to 0\), a complex with homology \(V\) in degree 0 and \(-1\).

The dg-algebra \(B\) has homology \(H^0(B)\) isomorphic to the path algebra of the quiver \(\bullet \longrightarrow \bullet\), \(H^{-1}(B) = k\) and \(H^1(B) = 0\) for \(i \neq 0, -1\).

The derived category of the hereditary algebra \(H^0(B)\) is not equivalent to \(D^b(A)\).

### 3.3. Graded of an abelian category

Let \(\mathcal{A}\) be an abelian \(k\)-linear artinian and noetherian category with finitely many simple objects up to isomorphism and \(S\) a complete set of representatives of isomorphism classes of simple objects. We assume \(\mathcal{A}\) is split, i.e., endomorphism rings of simple objects are isomorphic to \(k\). Let \(\mathcal{T} = D^b(\mathcal{A})\).

Let \(\text{gr}\mathcal{A}\) be the category with objects the objects of \(\mathcal{A}\) and where \(\text{Hom}_{\text{gr}\mathcal{A}}(M, N)\) is the graded vector space associated to the filtration of \(\text{Hom}_{\mathcal{A}}(M, N)\) given by \(\text{Hom}_{\mathcal{A}}(M, N)^i = \{f \mid \text{im} f \subseteq \text{rad}^i N\}\).

Given \(M \in \mathcal{A}\), let \(M_i = \text{rad}^i M, f_i : M_i \to M_{i-1}\) the inclusion, \(N_0 = M/M_1\) and \(\varepsilon_0 : M \to M/M_1\) the projection. This defines an object of \(\mathcal{F}\).

We obtain a functor \(\text{gr}\mathcal{A} \to \mathcal{F}\).

**Proposition 3.15.** The canonical functor \(\text{gr}\mathcal{A} \to \mathcal{F}\) is an equivalence.
Proof. The image of $\text{Hom}_A(N,N')$ in $\text{Hom}_A(N,N'_0)$ is isomorphic to the quotient of $\text{Hom}_A(N,N')$ by $\text{Hom}_A(N,\text{rad} N')$ and it follows that the functor is fully faithful.

Let us show that it is essentially surjective. Let $M \in \mathcal{F}$. Let $r \geq 0$ such that $M_{r+1} = 0$. Then, $M_i \rightarrow N_i$, has homology concentrated in degree 0 and is semi-simple. By induction on $-i$, it follows from the distinguished triangle $M_{i+1} \rightarrow M_i \rightarrow N_i \sim$ that $M_i$ has homology concentrated in degree 0.

Note that we have an exact sequence $0 \rightarrow H^0 M_{i+1} \rightarrow H^0 M_i \rightarrow H^0 N_i \rightarrow 0$. Since the canonical map $\text{Hom}(H^0 N_i,S) \rightarrow \text{Hom}(H^0 M_i,S)$ is bijective for any simple $S$, it follows that $H^0 N_i$ is the largest semi-simple quotient of $H^0 M_i$. So, $M_i \sim \text{rad}^i M_0$ and $M$ comes from an object of $\mathcal{A}$. \hfill $\Box$

4. Simple generators for stable categories

4.1. From equivalences. Let $k$ be a field and $A$ a split self-injective $k$-algebra with no projective simple module.

Let $B$ be another split self-injective $k$-algebra with no projective simple module, and let $F : B\text{-stab} \rightarrow A\text{-stab}$ be an equivalence of triangulated categories. Let $\mathcal{S}'$ be a complete set of representatives of isomorphism classes of simple $B$-modules. For $L \in \mathcal{S}'$, let $L'$ be an indecomposable $A$-module isomorphic to $F(L)$ in $A$-stab. Let $\mathcal{S} = \{L'\}_{L \in \mathcal{S}'}$. Then,

(i) $\text{Hom}_{A\text{-stab}}(S,T) = k^{d_{S,T}}$ for $S, T \in \mathcal{S}$. 
(ii) Every object $M$ of $A$-stab has a filtration $0 = M_r \rightarrow M_{r-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 = M$ such that the cone of $M_i \rightarrow M_{i-1}$ is isomorphic to an object of $\mathcal{S}$.

Note that (ii) is equivalent to

(ii') Given $M$ in $A$-mod, there is a projective module $P$ such that $M \oplus P$ has a filtration $0 = N_r \subset N_{r-1} \subset \cdots \subset N_1 \subset N_0 = M \oplus P$ with the property that $N_i/N_{i-1}$ is isomorphic (in $A$-mod) to an object of $\mathcal{S}$.

Linckelmann has shown the following [Li, Theorem 2.1 (iii)]:

Proposition 4.1. Assume that $F$ is induced by an exact functor $B\text{-mod} \rightarrow A\text{-mod}$.

If $\mathcal{S}$ consists of simple modules, then there is a direct summand of $F$ that is an equivalence $B\text{-mod} \sim A\text{-mod}$.

We deduce:

Corollary 4.2. Let $B_1, B_2$ be split self-injective algebras with no projective simple modules and $G_i : B_i\text{-mod} \rightarrow A\text{-mod}$ exact functors inducing stable equivalences. Assume $\mathcal{S}_1 = \mathcal{S}_2$ (up to isomorphism). Then, $B_1$ and $B_2$ are Morita equivalent.

So, if we assume in addition that $F$ comes from an exact functor $G$ between module categories, then $B$ is determined by $\mathcal{S}$, up to Morita equivalence.

The functor $G$ is isomorphic to $X \otimes_B -$ where $X$ is an $(A,B)$-bimodule. We can (and will) choose $G$ so that $X$ has no non-zero projective direct summand. Then, $G(L)$ is indecomposable for $L$ simple [Li, Theorem 2.1 (ii)], so $\mathcal{S} = \{G(L)\}_{L \in \mathcal{S}'}$, up to isomorphism.

Proposition 4.3. An $A$-module $M$ is in the image of $G$ if and only if there is a filtration $0 = M_r \subset M_{r-1} \subset \cdots \subset M_1 \subset M_0 = M$ such that $M_i/M_{i-1}$ is isomorphic to an object of $\mathcal{S}$.
Proof. Take $L$ a $B$-module. Then the image by $G$ of a filtration of $L$ whose successive quotients are simple provides a filtration as required.

Conversely, we proceed by induction on $r$. We have an exact sequence $0 \to G(N) \to M \to G(L) \to 0$ and a corresponding element $\zeta \in \text{Ext}_A^1(G(L), G(N))$. We have an isomorphism $\text{Ext}_B^1(L, N) \cong \text{Ext}_A^1(G(L), G(N))$ and we take $\zeta'$ to be the inverse image of $\zeta$ under this isomorphism. This gives an exact sequence $0 \to N \to M' \to L \to 0$, and hence an exact sequence $0 \to G(N) \to G(M') \to G(L) \to 0$ with class $\zeta$. It follows that $M \simeq G(M')$ and we are done. \hfill \qed

4.2. Filtrable objects.

4.2.1. Given two $A$-modules $M$ and $N$, we write $M \sim N$ to denote the existence of an isomorphism between $M$ and $N$ in $A$-stab. Given $f, g \in \text{Hom}_A(M, N)$, we write $f \sim g$ if $f - g$ is a projective map.

**Lemma 4.4.** Let $f, f' : M \to N$ be two surjective maps with $f \sim g$. Then there is $\sigma \in \text{Aut}_A(M)$ with $f' = f\sigma$ and $\sigma \sim \text{id}_M$.

Similarly, let $f, f' : N \to M$ be two injective maps with $f \sim g$. Then there is $\sigma \in \text{Aut}_A(M)$ with $f' = f\sigma$ and $\sigma \sim \text{id}_M$.

**Proof.** Let $L = \ker f$ and $L' = \ker f'$. Let $L = L_0 \oplus P$ and $L' = L'_0 \oplus P'$ with $P, P'$ projective and $L_0, L'_0$ without non-zero projective direct summands. We have an isomorphism $\alpha_0 \in \text{Hom}_{A, \text{stab}}(L_0, L'_0)$ in $A$-stab giving rise to an isomorphism of distinguished triangles in $A$-stab.

\[
\begin{array}{cccccc}
L_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Omega^{-1}L_0 \\
\alpha_0 \downarrow \sim & & \downarrow \sim & & \downarrow \sim \Omega^{-1}(\alpha_0) & \\
L'_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Omega^{-1}L'_0
\end{array}
\]

Let $\alpha_0 \in \text{Hom}_A(L_0, L'_0)$ lifting $\alpha_0$. This is an isomorphism. There is now a commutative diagram of $A$-modules, where the exact rows come from the elements of $\text{Ext}_A^1(N, L_0)$ and $\text{Ext}_A^1(N, L'_0)$ defined above:

\[
\begin{array}{cccccc}
0 & \longrightarrow & L_0 & \longrightarrow & M_0 & \longrightarrow & N & \longrightarrow & 0 \\
\alpha_0 \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
0 & \longrightarrow & L'_0 & \longrightarrow & M'_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}
\]

We have $M \simeq M_0 \oplus P \simeq M'_0 \oplus P'$, hence $P \simeq P'$. Let $\alpha : L \sim L'$ extending $\alpha_0$. Then there is $\sigma : M \sim M$ making the following diagram commute

\[
\begin{array}{cccccc}
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
\alpha \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim
\end{array}
\]

and we are done.

The second part of the lemma has a similar proof — it can also be deduced from the first part by duality. \hfill \qed
4.2.2.

**Hypothesis 2.** Let $\mathcal{S}$ be a finite set of indecomposable finitely generated $A$-modules such that $\text{Hom}_{A_{\text{stab}}}(S, T) = k^{S,T}$ for $S, T \in \mathcal{S}$.

An $\mathcal{S}$-filtration for an $A$-module $M$ is a filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ such that $M_i = M_i/M_{i+1}$ is in $\text{add}(\mathcal{S})$ for $0 \leq i \leq r - 1$.

We say that $M$ is filtrable if it admits an $\mathcal{S}$-filtration.

**Lemma 4.5.** Let $M$ be a non-projective filtrable $A$-module. Then there is $S \in \mathcal{S}$ such that $\text{Hom}_{A_{\text{stab}}}(M, S) \neq 0$ (resp. such that $\text{Hom}_{A_{\text{stab}}}(S, M) \neq 0.$)

**Proof.** Assume $\text{Hom}_{A_{\text{stab}}}(M, S) = 0$ for all $S \in \mathcal{S}$. Since $M$ is filtrable, it follows that $\text{End}_{A_{\text{stab}}}(M) = 0$, and hence $M$ is projective, which is not true. The second case is similar. 

**Lemma 4.6.** Let $M$ be a filtrable module and $S \in \mathcal{S}$. Given $f : M \to S$ non-projective, there is $g : M \to S$ surjective with filtrable kernel such that $f \sim g$. Similarly, given $f : S \to M$ non-projective, there is $g : S \to M$ injective with filtrable cokernel such that $f \sim g$.

**Proof.** We proceed by induction on the number of terms in a filtration of $M$. The result is clear if $M \in \mathcal{S}$.

Let $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} T \to 0$ be an exact sequence with $T \in \mathcal{S}$ and $N$ filtrable.

Assume first $f \alpha : N \to S$ is projective. Then there is $p : M \to S$ projective and $g : T \to S$ with $f - p = g\beta$. Since $g$ is not projective, it is an isomorphism. Consequently, $f - p$ is surjective and its kernel is isomorphic to $N$ by Lemma 4.4, so we are done.

Assume now $f \alpha : N \to S$ is not projective. By induction, there is $q : N \to S$ projective such that $f\alpha + q$ is surjective with filtrable kernel $N'$. Since $\alpha : N \to M$ is injective, there is a projective map $p : M \to S$ with $q = p\alpha$. Now, we have an exact sequence $0 \to N/N' \xrightarrow{\bar{\alpha}} M/\alpha(N') \to T \to 0$ and a non-projective surjection $f + p : M/\alpha(N') \to S$. Since $(f + p)\bar{\alpha} : N/N' \to S$ is an isomorphism, it follows that the kernel of the map $M/\alpha(N') \to S$ is isomorphic to $T$. Since $N'$ is filtrable, it follows that $\ker(f + p)$ is filtrable and we are done. The second assertion follows by duality. 

From Lemmas 4.4 and 4.6, we deduce :

**Lemma 4.7.** Let $S \in \mathcal{S}$ and let $M$ be a filtrable module.

If $f : M \to S$ be a surjective and non-projective map, then $\ker f$ is filtrable.

Similarly, if $g : S \to M$ is injective and non-projective, then $\text{coker} g$ is filtrable.

From Lemmas 4.5 and 4.6, we deduce :

**Lemma 4.8.** Let $M$ be filtrable non-projective. Then there is a submodule $S$ of $M$, with $S \in \mathcal{S}$, such that $M/S$ is filtrable and the inclusion $S \to M$ is not projective. Similarly, there is a filtrable submodule $N$ of $M$ such that $M/N \in \mathcal{S}$ and $M \to M/N$ is not projective.

**Proposition 4.9.** Let $M$ be an $A$-module with a decomposition $M \sim M_1 \oplus M_2$ in the stable category. If $M$ is filtrable then there is a decomposition $M = M_1 \oplus M_2$ such that $M_i$ is filtrable and $M_i \sim M_i'$. 


Proof. We can assume \( M \) is not projective, for otherwise the proposition is trivial. We prove the proposition by induction on the dimension of \( M \).

Let \( M = T_1 \oplus T_2 \oplus P \) with \( P \) projective, \( T_i \) without non-zero projective direct summand and \( T_i \sim M_i \). Denote by \( \pi : M \to T_1 \) the projection.

By Lemma 4.5, there is \( S \in S \) such that \( \text{Hom}_{A,\text{stab}}(M, S) \neq 0 \). Hence, \( \text{Hom}_{A,\text{stab}}(T_i, S) \neq 0 \) for \( i = 1 \) or \( i = 2 \). Assume for instance \( i = 1 \). Pick a non-projective map \( \alpha : T_1 \to S \). So, \( \alpha \pi : M \to S \) is not projective. By Lemma 4.6, there is a surjective map \( \beta : M \to S \) with \( \beta \sim \alpha \pi \) and \( N = \ker \beta \) filtrable. Then \( N \sim L \oplus T_2 \), where \( L \) is the kernel of \( \alpha + p : T_1 \oplus P_S \to S \) and \( p : P_S \to S \) is a projective cover of \( S \). By induction, we have \( N = N_1 \oplus N_2 \) with \( N_i \) filtrable and \( N_1 \sim L, N_2 \sim T_2 \). Now, the map \( S \to L[1] \) gives a map \( S \to N_1[1] \) (in \( A \)-stab). Let \( M_1 \) be the extension of \( S \) by \( N_1 \) corresponding to that map. Then \( M \simeq M_1 \oplus N_2 \), the modules \( M_1 \) and \( N_2 \) are filtrable, \( M_1 \sim M'_1 \), and \( N_2 \sim M'_2 \).

Let \( M \) be a filtrable module. We say that \( M \) has no projective remainder if there is no direct sum decomposition \( M = N \oplus P \) with \( P \neq 0 \) projective and \( N \) filtrable.

Lemma 4.10. Let \( M \) be a filtrable module with no projective remainder and let \( S \in S \).

For \( f : M \to S \) surjective, \( \ker f \) is filtrable if and only if \( f \) is non-projective.

For \( f : S \to M \) injective, \( \text{coker } f \) is filtrable if and only if \( f \) is non-projective.

Proof. Assume \( f \) is projective. Then there is a decomposition \( M = N \oplus P \) and \( f = (0, g) \) with \( P \) projective. Now, \( \ker f = N \oplus \ker g \). If \( \ker f \) is filtrable, then it follows from Lemma 4.9 that \( M \) has a non-zero projective submodule whose quotient is filtrable.

The converse is given by Lemma 4.7. The second part of the Lemma has a similar proof. \( \square \)

Lemma 4.11. Let \( M = M_0 \oplus M_1 \) with \( M \) and \( M_0 \) filtrable and such that \( M_0 \) has no projective remainder. Then \( M_1 \) is filtrable.

Proof. We proceed by induction on \( \dim M_0 \) — the result is clear for \( M_0 = 0 \). Assume \( M_0 \neq 0 \). Let \( f : M_0 \to S \) be a surjection with \( S \in S \) and \( \ker f \) filtrable. By Lemma 4.10, \( f \) is not projective. Then \( f' : M \xrightarrow{\text{can}} M_0 \xrightarrow{f} S \) is a non-projective surjection. By Lemma 4.7, \( \ker f' \) is filtrable. We have \( \ker f' = \ker f \oplus M_1 \) and we are done. \( \square \)

4.2.3. We now turn to filtrations by objects in \( \text{add}(S) \).

Lemma 4.12. Let \( M \) be a filtrable module and \( N \) a filtrable submodule of \( M \) such that \( M/N \in \text{add} S \). Then, \( N \) is minimal with these properties if and only if \( N \) has no projective remainder and the canonical map \( \text{Hom}_{A,\text{stab}}(M/N, S) \to \text{Hom}_{A,\text{stab}}(M, S) \) is surjective for every \( S \in S \).

Proof. Let \( N \) be a minimal filtrable submodule of \( M \) such that \( M/N \in \text{add} S \). Denote by \( i : N \to M \) the injection and \( p : M \to M/N \) the quotient map.

Let \( S \in S \). Fix \( f_1, \ldots, f_r : M/N \to S \) such that \( \sum f_i : M/N \to S^r \) is surjective and \( \ker \sum f_i \) has no direct summand isomorphic to \( S \). Let \( T \) be the subspace of \( \text{Hom}_{A,\text{stab}}(M, S) \) generated by \( f_1p, \ldots, f_rp \). Assume this is a proper subspace, so there is \( f' : M \to S \) whose image in \( \text{Hom}_{A,\text{stab}}(M, S) \) is not in \( T \). Then \( f'i : N \to S \) is not projective, hence there is a projective \( q : N \to S \) such that \( f'i + q \) is surjective and has filtrable kernel \( N' \) (Lemma 4.6). There is \( q' : M \to S \) projective such that \( q = q'i \). Now, \( M/N' \simeq M/N \oplus S \) and this contradicts the minimality of \( N \). It follows that the canonical map \( \text{Hom}_{A,\text{stab}}(M/N, S) \to \text{Hom}_{A,\text{stab}}(M, S) \) is surjective. Assume \( N = N' \oplus P \) with \( N' \) filtrable with no projective remainder and \( P \) projective.
By Lemma 4.11, $P$ is filtrable. We have $M/N' \simeq M/N \oplus P$. Since $M/N$ is a maximal quotient of $M$ in $\text{add}(S)$ and $P$ is filtrable, it follows that $P = 0$.

Conversely, take $f : N \to S$ surjective with filtrable kernel such that the extension of $M/N$ by $S$ splits. Then $f$ lifts to $M \to S$ and it is not projective by Lemma 4.10. This contradicts the surjectivity of $\text{Hom}_{A\text{-stab}}(M/N, S) \to \text{Hom}_{A\text{-stab}}(M, S)$. Consequently, $N$ is minimal. □

**Lemma 4.13.** Let $M$ be a filtrable $A$-module with no projective remainder.

Let $f : M \to L$ be a surjection with $L \in \text{add} S$. Then $\text{ker } f$ is filtrable if and only if the canonical map $\text{Hom}_{A\text{-stab}}(L, S) \to \text{Hom}_{A\text{-stab}}(M, S)$ is injective for all $S \in S$.

**Proof.** Note that the canonical map $\text{Hom}_{A\text{-stab}}(L, S) \to \text{Hom}_{A\text{-stab}}(M, S)$ is injective if and only if, given $p : L \to S$ surjective with $S \in S$, $pf$ is not projective.

Assume $\text{ker } f$ is filtrable. Let $p : L \to S$ be a surjective map with $S \in S$. Then $\text{ker } pf$ is filtrable, hence $pf$ is not projective (Lemma 4.10).

Let us now prove the converse by induction on the dimension of $M$. Assume that given $p : L \to S$ surjective with $S \in S$, then $pf$ is not projective. Pick $p : L \to S$ surjective and let $L' = \text{ker } p$. Let $M' = \text{ker } pf$. Then $f$ induces a surjection $f' : M' \to L'$ and we have $L' \in \text{add} S$ (since $p$ is split). Let $p' : L' \to T$ be a surjective map with $T \in S$. Fix a left inverse $\sigma : L \to L'$ to the inclusion $L' \to L$.

If $S \neq T$, then $\text{Hom}_{A\text{-stab}}(S, T) = 0$, and hence $p'\sigma f$ doesn’t factor through $S$ in the stable category. On the other hand, if $S = T$ then $pf$ and $p'\sigma f$ define linearly independent elements of $\text{Hom}_{A\text{-stab}}(M, S)$. Consequently, $p'\sigma f$ doesn’t factor through $S$ in the stable category. It follows that $p'f'$ is not projective. By Lemma 4.7, $M'$ is filtrable. By induction, it follows that $\text{ker } f'$ is filtrable and we are done. □

**Proposition 4.14.** Let $M$ be a filtrable $A$-module with no projective remainder.

Let $N$ be a minimal filtrable submodule of $M$ such that $M/N \in \text{add} S$. Then there is an isomorphism

$$M/N \cong \bigoplus_{S \in S} S \otimes \text{Hom}_{A\text{-stab}}(M, S)$$

that induces the canonical map $M \to \bigoplus_{S \in S} S \otimes \text{Hom}_{A\text{-stab}}(M, S)$ in the stable category.

Given $\tau \in \text{Aut}(N)$ such that $\tau \sim \text{id}_N$, there is $\sigma \in \text{Aut}(M)$ with $\sigma \sim \text{id}_M$ and $\sigma|N = \tau$.

Let $N'$ be a minimal filtrable submodule of $M$ such that $M/N' \in \text{add} S$. Then there is $\sigma \in \text{Aut}(M)$ such that $N' = \sigma(N)$ and $\sigma \sim \text{id}_M$. 

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**Diagram:**

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In the diagram, $M/N' \to \cdots \to S$ is a minimal filtrable submodule of $M$, and $\sigma$ is a left inverse to $N' \to M$.
Proposition 4.15. Let $M \sigma$ is stably the identity. We prove this lemma by induction on the dimension of $M$

Proof. The first part of the proposition follows from Lemmas 4.12 and 4.13.

Let $\tau \in \Aut(N)$ such that $\tau = \id_N + p$ with $p: N \to N$ projective. Then there is a projective map $q: M \to N$ with $p = qi$. Let $\sigma = \id_N + q$. Then $\sigma|N = \tau$. Now, we have a commutative diagram

$$
\begin{array}{c}
0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0 \\
\tau \sim \sigma \downarrow \quad \downarrow \quad \downarrow \id \\
0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0
\end{array}
$$

and hence $\sigma$ is an automorphism of $M$.

Let $N'$ be a minimal filtrable submodule of $M$ such that $M/N' \in \add S$. Then we have shown that $M/N \xrightarrow{\sim} M/N'$ and that via such an isomorphism, the maps $M \to M/N$ and $M \to M/N'$ are stably equal. Now, Lemma 4.4 shows there is $\sigma \in \Aut(M)$ with $N' = \sigma(N)$ and $\sigma \sim \id_M$.

Let $M$ be filtrable. An $S$-radical filtration of $M$ is a filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ such that $M_i$ is a minimal filtrable submodule of $M_{i-1}$ with $M_{i-1}/M_i \in \add S$.

Proposition 4.15. Let $M$ be a filtrable $A$-module with no projective remainder. Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ and $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_0 = M$ be two $S$-radical filtrations of $M$. Then, $r = r'$ and there is an automorphism of $M$ that swaps the two filtrations and that is stably the identity.

Proof. We prove this lemma by induction on the dimension of $M$. By Proposition 4.14, there is $\sigma \in \Aut(M)$ such that $\sigma(M'_i) = M_i$ and $\sigma \sim \id_M$. Now, by induction, we have $r = r'$ and there is $\tau \in \Aut(M_i)$ such that $\tau \sigma(M'_i) = M_i$ for $i > 0$ and $\tau \sim \id_M$. By Proposition 4.14, there is $\tau' \in \Aut(M)$ such that $\tau'_{M'_i} = \tau_i$ and $\tau' \sim \id_M$. Now, $\tau' \sigma$ sends $M'_i$ onto $M_i$.

Remark 4.16. A filtrable projective module can have two $S$-radical filtrations with non-isomorphic layers.

Consider $A = k\mathfrak{A}_4$, the group algebra of the alternating group of degree 4 and assume $k$ has characteristic 2 and contains a cubic root of 1. Let $B$ be the principal block of $k\mathfrak{A}_5$. Then, the restriction functor is a stable equivalence between $B$ and $A$. Let $S$ be the set of images of the simple $B$-modules. Denote by $k$ the trivial $A$-module and by $k_+, k_-$ the non-trivial simple $A$-modules. Then $S = \{k, S_+, S_-\}$ where $S_\varepsilon$ is a non-trivial extension of $k_\varepsilon$ by $k_{-\varepsilon}$. Let $P$ and $P'$ be the two projective indecomposable $B$-modules that don’t have $k$ as a quotient. Then $\Res_{A_i} P \simeq \Res_{A_i} P'$. This projective module has two $S$-radical filtrations with non-isomorphic layers: one coming from the radical filtration of $P$ and one coming from the radical filtration of $P'$.

While $S$-radical filtrations are not unique in general for filtrable modules with a projective remainder, there are some cases where uniqueness still holds:

Proposition 4.17. Assume $A$ is a symmetric algebra. Let $0 \to S \to M \to T \to 0$ and $0 \to S' \to M \to T' \to 0$ be two exact sequences with $S, S', T, T' \in S$. Assume that the sequences don’t both split. Then there is an automorphism of $M$ swapping the two exact sequences.

Proof. If $M$ is non-projective, then this is a consequence of Proposition 4.14.
Assume $M$ is projective. Since $A$ is symmetric, we have a non-projective map $T \simeq \Omega^{-1}S \to S$. It follows that $S = T$. Similarly, $T' = S'$. We have exact sequences
\[
0 \to \text{Hom}(S', S) \to \text{Hom}(S', M) \to \text{Hom}(S', S) \to \text{Ext}^1(S', S) \to 0
\]
\[
0 \to \text{Hom}(S', S') \to \text{Hom}(S', M) \to \text{Hom}(S', S') \to \text{Ext}^1(S', S') \to 0
\]
We have $\Omega^{-1}S' \simeq S'$, and hence $\dim \text{Ext}^1(S', S') = 1$. Consequently, $\dim \text{Hom}(S', M)$ is an odd integer. It follows that $\text{Ext}^1(S', S) \neq 0$, hence $\text{Hom}_{A,\text{stab}}(S', S) \neq 0$, so $S' = S$ and we are done by Lemma 4.4.

\[\square\]

**Lemma 4.18.** Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ be a filtration of $M$ with $M_{i-1}/M_i \in \text{add}S$.

(i) If $M$ has no projective remainder, then $M_i$ has no projective remainder, for all $i$.

(ii) If the filtration is an $S$-radical filtration, then $M_i$ has no projective remainder for $i \geq 1$.

**Proof.** Consider an exact sequence $0 \to N \oplus P \to M \to L \to 0$ of filtrable modules with $P$ projective and $N$ filtrable. Then there is an extension $M'$ of $L$ by $N$ such that $M = M' \oplus P$ and $M'$ is filtrable. The first part of the lemma follows.

Assume now the filtration is an $S$-radical filtration. Assume for some $i \geq 1$, we have $M_i = N \oplus P$ with $N$ filtrable with no projective remainder and $P$ projective and filtrable (Lemma 4.11). Then, $M = M' \oplus P$ with $P$ filtrable by (i). There is an exact sequence $0 \to L \to P \to S \to 0$ with $S \in S$ and $L$ filtrable. Now, the canonical surjection $M' \oplus P \to M/M_1 \oplus S$ has filtrable kernel and this contradicts the minimality of $M_1$. $\square$

**Proposition 4.19.** Let $M_1$ and $M_2$ be two filtrable $A$-modules with no projective remainder. If $M_1 \sim M_2$, then $M_1 \simeq M_2$.

**Proof.** We prove the proposition by induction on $\min(\dim M_1, \dim M_2)$. Fix an isomorphism $\phi$ from $M_2$ to $M_1$ in the stable category. Let $X = \bigoplus_{S \in S} S \otimes \text{Hom}_{A,\text{stab}}(M_1, S)$ and $g_1 \in \text{Hom}_{A,\text{stab}}(M_1, X)$ be the canonical map. Let $g_2 = g_1 \phi$. By Propositions 4.14 and 4.15, there are exact sequences
\[
0 \to N_1 \to M_1 \xrightarrow{f_1} X \to 0 \quad \text{and} \quad 0 \to N_2 \to M_2 \xrightarrow{f_2} X \to 0
\]
with the image of $f_i$ in the stable category equal to $g_i$. So, there is an isomorphism from $N_2$ to $N_1$ in the stable category compatible with $\phi$. By Lemma 4.18, $N_1$ and $N_2$ have no projective remainder. By induction, we deduce that there is an isomorphism $N_2 \xrightarrow{\sim} N_1$ lifting the stable isomorphism. So, $M_1$ and $M_2$ are extensions of isomorphic modules, with the same class in $\text{Ext}^1$, hence are isomorphic. $\square$

4.3. Generators and reconstruction.

4.3.1. We assume from now on that

**Hypothesis 3.** $S$ satisfies Hypothesis 2 and given $M \in A\text{-mod}$, there is a projective $A$-module $P$ such that $M \oplus P$ is filtrable.

**Proposition 4.20.** Let $S \in S$. Let $P_S \to S$ be a projective cover of $S$ and $P$ minimal projective such that $\Omega S \oplus P$ is filtrable. Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = \Omega S \oplus P$ be an $S$-radical filtration.

Then $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 \subseteq P_S \oplus P$ is an $S$-radical filtration.

If $A$ is symmetric, then $M_{r-1} \simeq S$. 
Proof. Let $f_1 : P_S \to S$ be a surjective map and $f = (f_1, 0) : P_S \oplus P \to S$. Let $T \in S$ and $g : P_S \oplus P \to T$ such that we have an exact sequence $0 \to L \to P_S \oplus P \xrightarrow{f+g} S \oplus T \to 0$ with $L$ filtrable.

We have a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & L \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Omega S \oplus P \\
\downarrow & (0,\text{id}) & \downarrow \\
0 & \longrightarrow & T
\end{array}
\]

The surjection $\Omega S \oplus P \to T$ is projective and has filtrable kernel. From Lemma 4.10, we get a contradiction to the minimality of $P$. It follows that $\Omega S \oplus P$ is a minimal submodule of $P_S \oplus P$ such that the quotient is in $\text{add} \ S$.

We have $\text{Hom}_{A\text{-stab}}(T, \Omega S) \simeq \text{Hom}_{A\text{-stab}}(S, T)^\ast$, since $A$ is symmetric. Now, $\text{Hom}_{A\text{-stab}}(M_{r-1}, \Omega S \oplus P) \neq 0$ by Lemma 4.10. The second part of the proposition follows. \qed

Let $M$ and $N$ be two $A$-modules with filtrations $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ $0 = N_s \subseteq N_{s-1} \subseteq \cdots \subseteq N_0 = N$. Let $\text{Hom}_A^f(M, N)$ be the subspace of $\text{Hom}_A(M, N)$ of filtered maps (i.e., those $g$ such that $g(M_i) \subseteq N_i$). We put $\tilde{M}_i = M_i/M_{i+1}$. We denote by $\phi_i$ the composition of canonical maps $\phi_i : \text{Hom}_A^f(M, N) \to \text{Hom}_A(\tilde{M}_i, \tilde{N}_i) \to \text{Hom}_{A\text{-stab}}(\tilde{M}_i, \tilde{N}_i)$.

We view $N' = N_i$ as a filtered module with the induced filtration $0 = N_{s-i} \subseteq N_{s-i-1} = N_{s-1} \subseteq \cdots \subseteq N'_i = N_{i+1} \subseteq N'_0 = N'$.

**Lemma 4.21.** Let $M$ be a filtrable $A$-module with an $\mathcal{S}$-radical filtration and $N$ be a filtrable $A$-module with an $\mathcal{S}$-filtration. Let $f \in \text{Hom}_A^f(M, N)$ with $\phi_0(f) = 0$. Then $\phi_i(f) = 0$ for all $i$.

**Proof.** The map $\bar{f}_0 : \tilde{M}_0 \to \tilde{N}_0$ induced by $f$ is projective. So there is a projective module $P$ and a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{P} & N \\
\downarrow & & \downarrow \\
\tilde{M}_0 & \xrightarrow{\bar{f}_0} & \tilde{N}_0
\end{array}
\]

Let $p$ be the composition $p : M \to \tilde{M}_0 \to P \to N$. Then $f - p \sim f$, $f - p$ and $f$ have the same restriction to $M_1$, and $(f - p)_{|_{M_1}} = 0$. Consequently it is enough to prove the lemma in the case where $\bar{f}_0 = 0$.

From now on, we assume $\bar{f}_0 = 0$. Assume the map $\bar{f}_1 : \tilde{M}_1 \to \tilde{N}_1$ is not projective. So there is $S \in \mathcal{S}$ and a (split) surjection $g : \tilde{N}_1 \to S$ such that $g \bar{f}_1 : M_1 \to S$ is not projective. Let $s : S \to \tilde{M}_1$ be a right inverse to $g$, and let $L$ be the kernel of $g \bar{f}_1$.

We have an exact sequence $0 \to L \to M/M_2 \xrightarrow{\text{can},gf} \tilde{M}_0 \oplus S \to 0$. So the inverse image of $L$ in $M_1$ is a filtrable submodule of $M$ with quotient isomorphic to $\tilde{M}_0 \oplus S$. This contradicts the fact that $M_1$ is a minimal filtrable submodule of $M$ such that $M/M_1 \in \text{add} \ S$. So $\bar{f}_1$ is projective; i.e., $\phi_1(f) = 0$.\qed
We now prove by induction that $\phi_i(f) = 0$ for all $i$. Assume $\phi_d(f) = 0$. Then, we apply the result above to the filtered modules $M_d$ and $N_d$ to get $\phi_{d+1}(f) = 0$. 

4.3.2. We define a category $\mathcal{G}$ as follows:

- Its objects are $A$-modules together with a fixed $\mathcal{S}$-radical filtration.
- We define $\text{Hom}_\mathcal{G}(M, N)_i$ as the image of $\text{Hom}_A^i(M, N_i)$ in $\text{Hom}_{\mathcal{A}_{\text{stab}}}(\overline{M}_0, \overline{N}_i)$. We put $\text{Hom}_\mathcal{G}(M, N) = \oplus_i \text{Hom}_\mathcal{G}(M, N)_i$.
- Let $f \in \text{Hom}_\mathcal{G}(M, N)_i$ and $g \in \text{Hom}_\mathcal{G}(L, M)_j$. Let $\tilde{f}: M \to N_i$ be a filtered map lifting $f$. It induces a map $\phi_j(\tilde{f}) \in \text{Hom}_{\mathcal{A}_{\text{stab}}}(M_j, \overline{N}_{i+j})$ independent of the choice of $\tilde{f}$ (Lemma 4.21). We define the product $fg$ to be $\phi_j(\tilde{f}) \circ \phi_0(g)$.

Given $S \in \mathcal{S}$, let $P_S \to S$ be a projective cover of $S$ and $Q_S$ projective minimal such that $\Omega S \oplus Q_S$ is filtrable. Fix a radical filtration of $P_S \oplus Q_S$ with first term $\Omega S \oplus Q_S$.

Let $M = \oplus_{S \in \mathcal{S}} (P_S \oplus Q_S)$. This comes with an $\mathcal{S}$-radical filtration. We have constructed a $\mathbb{Z}_{\geq 0}$-graded $k$-algebra $\text{End}_\mathcal{G}(M)$.

The following Lemma is clear.

Lemma 4.22. Let $\mathcal{S}$ be a complete set of representatives of isomorphism classes of simple $A$-modules. Then we have an equivalence $\text{gr}(A_{\text{mod}}) \xrightarrow{\sim} \mathcal{G}$. If $A$ is basic, then $\text{End}_\mathcal{G}(M)$ is isomorphic to the graded algebra associated with the radical filtration of $A$.

We have now obtained our partial reconstruction result:

Theorem 4.23. Let $B$ be a selfinjective algebra with no simple projective module. Let $M$ be an $(A, B)$-bimodule inducing a stable equivalence and having no projective direct summand. Let $\mathcal{S} = \{ M \otimes_B L \}$ where $L$ runs over a complete set of representatives of isomorphism classes of simple $B$-modules.

Then, there is an equivalence $\text{gr}(B_{\text{mod}}) \xrightarrow{\sim} \mathcal{G}$. If $B$ is basic, there is an isomorphism between the graded algebra associated with the radical filtration of $B$ and $\text{End}_\mathcal{G}(M)$.

4.3.3. The category $\mathcal{G}$ can be constructed directly as in §3.1, using only the stable category with its triangulated structure.

Proposition 4.24. Let $M$ be a module with an $\mathcal{S}$-filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$. This is an $\mathcal{S}$-radical filtration if and only if

- $\text{Hom}_{\mathcal{A}_{\text{stab}}}(M_i/M_{i+1}, S) \to \text{Hom}_{\mathcal{A}_{\text{stab}}}(M_i, S)$ is an isomorphism for all $S \in \mathcal{S}$ and $i > 0$,
- $\text{Hom}_{\mathcal{A}_{\text{stab}}}(M_0/M_1, S) \to \text{Hom}_{\mathcal{A}_{\text{stab}}}(M_0, S)$ is surjective for all $S \in \mathcal{S}$, and
- $M_i$ has no projective remainder for $i > 0$.

Assume the filtration is an $\mathcal{S}$-radical filtration. Then $M$ has no projective remainder if and only if $\text{Hom}_{\mathcal{A}_{\text{stab}}}(M_0/M_1, S) \to \text{Hom}_{\mathcal{A}_{\text{stab}}}(M_0, S)$ is an isomorphism.

Proof. Let $M$ be a module with an $\mathcal{S}$-radical filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$. The canonical map $\text{Hom}_{\mathcal{A}_{\text{stab}}}(M_i/M_{i+1}, S) \to \text{Hom}_{\mathcal{A}_{\text{stab}}}(M_i, S)$ is surjective for all $S \in \mathcal{S}$, by Lemma 4.12. Note that $M_i$ has no projective remainder for $i > 0$, by Lemma 4.18. It follows that the canonical map $\text{Hom}_{\mathcal{A}_{\text{stab}}}(M_i/M_{i+1}, S) \to \text{Hom}_{\mathcal{A}_{\text{stab}}}(M_i, S)$ is an isomorphism for all $S \in \mathcal{S}$ (Lemma 4.13).

Let us now prove the other implication. Since $M_i$ has no projective remainder for $i > 0$, it follows from Lemma 4.12 that $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1$ is an $\mathcal{S}$-radical filtration of $M_1$. 


Assume the filtration is an $S$-radical filtration. If $M$ has no projective remainder, then $\text{Hom}_{A\text{-stab}}(M_0/M_1, S) \to \text{Hom}_{A\text{-stab}}(M_0, S)$ is injective by Lemma 4.13.

Assume now that $\text{Hom}_{A\text{-stab}}(M/M_1, S) \to \text{Hom}_{A\text{-stab}}(M, S)$ is bijective. Assume $M = M' \oplus P$ with $M'$ filtrable and $P$ projective. We have $\text{Hom}_{A\text{-stab}}(M/M_1, S) \xrightarrow{\sim} \text{Hom}_{A\text{-stab}}(M, S) \xrightarrow{\sim} \text{Hom}_{A\text{-stab}}(M', S)$. There is a surjective map $g : M' \to M/M_1$ with filtrable kernel such that the composition $M \xrightarrow{\text{can}} M' \xrightarrow{g} M/M_1$ is equal to the canonical map $M \to M/M_1$ in the stable category, by Proposition 4.14. By Lemma 4.4, we have $M_1 \simeq \ker g \oplus P$. Since $M_1$ has no projective remainder by the first part of the proposition, we get $P = 0$, hence $M$ has no projective remainder.

Let $\mathcal{T} = A\text{-stab}$. Note that $S$ is determined by its image in $\mathcal{T}$ and it satisfies Hypothesis 3 if and only if $\text{Hom}_\mathcal{T}(S, T) = k^{\delta_{ST}}$ for all $S, T \in \mathcal{S}$ and every object of $\mathcal{T}$ is an iterated extension of objects of $\mathcal{S}$.

We have a functor $\mathcal{G} \to \mathcal{F}$: it sends a module $M$ with an $S$-radical filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ to $\cdots \to 0 \to M_{r-1} \to \cdots \to M_1 \to M \to M/M_1$ (cf Proposition 4.24).

**Proposition 4.25.** The canonical functor $\mathcal{G} \xrightarrow{\sim} \mathcal{F}$ is an equivalence.

**Proof.** The functor is clearly fully faithful.

Start with $0 = N_r \xrightarrow{f_r} N_{r-1} \to \cdots \to N_1 \xrightarrow{f_1} N_0 \xrightarrow{\varepsilon_0} M_0$. Adding a projective direct summand to the $N_i$’s, we can lift the maps $f_i$ to maps that are injective in the module category and such that the successive quotients have no projective direct summands. So we have a filtration $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_1 \subseteq M'_0$ such that $M'_i/M'_i$ is stably isomorphic to a direct sum of objects of $\mathcal{S}$. Since it has no projective summand, it is actually isomorphic to a sum of objects of $\mathcal{S}$; i.e., we have an $S$-filtration. Consider $i$ maximal such that $M'_i$ has a projective remainder. Then $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_i$ is an $S$-radical filtration by Proposition 4.24 (first part). The second part of Proposition 4.24 shows now that $M'_i$ has no projective remainder, a contradiction. So the filtration is an $S$-filtration.

**References**


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