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Minimising Dirichlet eigenvalues on cuboids of unit measure

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K. Gittins 2

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Abstract

We consider the minimisation of Dirichlet eigenvalues \( \lambda_k \), \( k \in \mathbb{N} \), of the Laplacian on cuboids of unit measure in \( \mathbb{R}^3 \). We prove that any sequence of optimal cuboids in \( \mathbb{R}^3 \) converges to a cube of unit measure in the sense of Hausdorff as \( k \to \infty \). We also obtain an upper bound for that rate of convergence.

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1 Introduction.

The eigenvalues of the Laplacian have been the object of intensive study over the last century. Of particular interest are related shape optimisation problems. For \( k \in \mathbb{N} \), the goal is to optimise the \( k \)’th eigenvalue of the Laplacian with boundary conditions over a collection of open sets in \( \mathbb{R}^m \). This collection satisfies geometric constraints, such as fixed Lebesgue measure or fixed perimeter.

For an open set \( \Omega \subset \mathbb{R}^m \), \( m \geq 2 \), of finite Lebesgue measure \( |\Omega| \), we let \( \lambda_k(\Omega) \), \( k \in \mathbb{N} \), denote the Dirichlet eigenvalues of the Laplacian on \( \Omega \) which are strictly positive, arranged in non-decreasing order and counted with multiplicity:

\[
\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \cdots
\]

This sequence accumulates at \( +\infty \).

We consider the following minimisation problem:

\[
\lambda^*_k(m) := \inf \{ \lambda_k(\Omega) : \Omega \text{ open in } \mathbb{R}^m, |\Omega| = c \}.
\]

It was shown by Faber and Krahn that among all open sets in \( \mathbb{R}^m \) of measure \( c \), the ball of measure \( c \) minimises the first Dirichlet eigenvalue, see [15]. Krahn and Szegö proved that, among all open sets in \( \mathbb{R}^m \) of measure \( c \), the second Dirichlet eigenvalue is minimised by the union of two disjoint balls of measure \( \frac{c}{2} \) each, see [15]. For \( k \geq 3 \), the existence of an open set of prescribed measure which minimises the \( k \)’th Dirichlet eigenvalue remains unresolved to date. However, in the class of quasi-open sets of prescribed measure, it was shown by Bucur in [7] that a minimiser does exist and that such a minimiser is bounded and has finite perimeter. Independently, Mazzoleni and Pratelli proved the existence of a minimiser in [17] in the collection of quasi-open sets. For any lower semi-continuous,
increasing function of the first $k$ Dirichlet eigenvalues, they proved the existence of a minimiser which is bounded in terms of $k$ and $m$ independently of the function. It was shown in [5] that for $k \leq m + 1$, any bounded minimiser of $\lambda_k(\Omega)$ has at most $\min\{7, k\}$ components.

No optimal domains are known for $\lambda_k$ with $k \geq 3$. In particular, the conjecture that if $m = 2$, then $\lambda_3(\Omega)$ is bounded from below by the third eigenvalue of the disc with the same measure as $\Omega$ is open. There are no obvious candidates for minimisers of $\lambda_k$ with $k \geq 5$ in any dimension $m \geq 2$. Even for $m = 2$, minimisers need not be discs or disjoint unions of discs, see [21]. Furthermore, it was shown in [6] that for $k \geq 5$, $\lambda_k(\Omega)$ cannot be minimised by a disc or a disjoint union of discs. The numerical investigation [1] suggests that for some values of $k$ the minimisers may not have any symmetries.

Pólya’s conjecture for Dirichlet eigenvalues asserts that for all bounded, open sets $\Omega \subset \mathbb{R}^m$, $\lambda_k(\Omega) \geq 4\pi^2(\omega_m[\Omega])^{-2/m}k^{2/m}$, where $\omega_m$ denotes the measure of a ball in $\mathbb{R}^m$ of radius 1. It was shown in [11] that Pólya’s conjecture is equivalent to $\lambda_k^c(m)$ being asymptotically equal to $4\pi^2(\omega_m c)^{-2/m}k^{2/m}$ as $k \to \infty$.

It is also interesting to consider the optimisation of the eigenvalues of the Laplacian subject to other geometric constraints, such as fixed perimeter. For the Dirichlet eigenvalues, existence of a minimiser in the class of open sets in $\mathbb{R}^m$ of finite Lebesgue measure and prescribed perimeter was shown in [12]. Moreover, it was shown there that any minimiser is bounded and connected, and regularity results for the boundary were also obtained. Bucur and Freitas, [9], showed that any sequence of minimisers of $\lambda_k$ in $\mathbb{R}^2$ with perimeter $\ell$ converges in the sense of Hausdorff to the disc of perimeter $\ell$ as $k \to \infty$. They also showed that if the collection of admissible sets is restricted to the collection of $n$-sided, convex, planar polygons of perimeter $\ell$, then any sequence of minimisers converges to the regular $n$-sided polygon of perimeter $\ell$ as $k \to \infty$. For $m \geq 2$, other constraints were considered in [4], including perimeter and moment of inertia, subject to an additional convexity constraint. Further results for the Dirichlet eigenvalues were obtained in [3], [8], [5], [9] and [4]. Some of the results of [3] follow directly from those in [4], while the results of [12] supersede those of [8].

Recently, Antunes and Freitas considered the problem of minimising $\lambda_k$ over all planar rectangles of unit measure, [2]. In Theorem 2.1 of [2], they showed that any sequence of minimising rectangles for the Dirichlet eigenvalues converges to the unit square in the sense of Hausdorff as $k \to \infty$.

In Theorem 1.1 below we obtain the corresponding 3-dimensional result for the Dirichlet eigenvalues of the Laplacian on cuboids in $\mathbb{R}^3$ of unit measure. In addition we obtain an estimate for the rate of convergence. Let $R_{a_1,a_2,a_3}$ denote a cuboid in $\mathbb{R}^3$ of side-lengths $a_1, a_2, a_3$ such that $a_1a_2a_3 = 1$ and $a_1 \leq a_2 \leq a_3$.

$$R_{a_1,a_2,a_3} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < a_1, 0 < x_2 < a_2, 0 < x_3 < (a_1a_2)^{-1}, a_1 \leq a_2 \leq a_3\}. \quad (1.1)$$

We prove the following.

**Theorem 1.1**

(i) Let $k \in \mathbb{N}$. The variational problem

$$\lambda_k^c := \inf \{\lambda_k(R_{a_1,a_2,a_3}) : a_1 \leq a_2 \leq a_3\}$$

has a minimising cuboid $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ with side-lengths $a_{1,k}^* \leq a_{2,k}^* \leq a_{3,k}^*$, such that $a_{1,k}^*a_{2,k}^*a_{3,k}^* = 1$.

(ii)

$$a_{3,k}^* \leq 1 + O(k^{-(2-\beta)/6}), \quad k \to \infty, \quad (1.2)$$

where $\beta$ is an exponent of the remainder in

$$\# \{(i_1, i_2, i_3) \in \mathbb{Z}^3 : i_1^2 + i_2^2 + i_3^2 \leq R^2\} - \frac{4\pi}{3} R^3 = O(R^3), \quad R \to \infty.$$

Furthermore, any sequence of optimal cuboids $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ converges to the unit cube in $\mathbb{R}^3$ in the sense of Hausdorff as $k \to \infty$. 

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The best known estimate to date is that for any \( \epsilon > 0, \beta = \frac{21}{16} + \epsilon \), see [14]. Hence (1.2) holds for \( \beta = \frac{21}{16} + \epsilon, \epsilon > 0 \). The conjecture for the optimal remainder is \( \beta = 1 + \epsilon, \epsilon > 0 \). See [10].

A heuristic explanation for this asymptotic shape result is the following (see also [2]). For any cuboid \( R \) in \( \mathbb{R}^3 \) with measure \( |R| \) and perimeter \( \text{Per}(R) \), one has that

\[
\lambda_k(R) = \frac{6\pi^2 k}{|R|^{2/3}} + \frac{(3\pi^5)^{1/3}\text{Per}(R)k^{1/3}}{2^{5/3}|R|^{4/3}} + o(k^{1/3}), \ k \to \infty. \tag{1.3}
\]

So if \( |R| = 1 \) then (1.3) suggests that the cuboid that minimises \( \lambda_k(R) \), \( k \to \infty \), is the one with minimal perimeter, i.e. the unit cube.

The Dirichlet eigenvalues of the Laplacian on a cuboid \( R_{a_1,a_2,a_3} \) (as in (1.1)) are given by

\[
\frac{\pi^2 i_1^2}{a_1^2} + \frac{\pi^2 i_2^2}{a_2^2} + \frac{\pi^2 i_3^2}{a_3^2}, \ i_1,i_2,i_3 \in \mathbb{N}. \tag{1.4}
\]

By listing these in non-decreasing order including multiplicities, the \( k \)’th Dirichlet eigenvalue on \( R_{a_1,a_2,a_3} \), \( \lambda_k(R_{a_1,a_2,a_3}) \), is the \( k \)’th item of this list. In the table below we list the minimising cuboids for the first few Dirichlet eigenvalues.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \lambda_k )</th>
<th>( a_{1,k}, a_{2,k}, a_{3,k} )</th>
<th>Minimising modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 3\pi^2 )</td>
<td>( 1,1,1 )</td>
<td>( 1,1,1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 3 \cdot 2^{2/3} \pi^2 )</td>
<td>( 2^{-1/3}, 2^{-1/3}, 2^{2/3} )</td>
<td>( 1,1,2 )</td>
</tr>
<tr>
<td>3</td>
<td>( 3 \cdot 2^{-2/3} \pi^2 )</td>
<td>( \left( \frac{1}{2} \right)^{1/3}, \left( \frac{2}{3} \right)^{1/6}, \left( \frac{1}{3} \right)^{1/6} )</td>
<td>( 1,2,1 )</td>
</tr>
<tr>
<td>4</td>
<td>( 6\pi^2 )</td>
<td>( 1,1,1 )</td>
<td>( 2,1,1 )</td>
</tr>
<tr>
<td>5</td>
<td>( 3^{2/3} \pi^2 )</td>
<td>( 3^{-1/3}, 3^{-1/3}, 3^{2/3} )</td>
<td>( 1,1,3 )</td>
</tr>
<tr>
<td>6</td>
<td>( 3 \cdot 2^{4/3} \pi^2 )</td>
<td>( 2^{-2/3}, 2^{1/3}, 2^{1/3} ) or ( 2^{-2/3}, 2^{-2/3}, 2^{4/3} )</td>
<td>( 1,2,2 ) or ( 1,1,4 )</td>
</tr>
<tr>
<td>7</td>
<td>( 3 \cdot 5^{2/3} \pi^2 )</td>
<td>( \left( \frac{5}{2} \right)^{1/6}, \left( \frac{2}{3} \right)^{1/6}, 2 \cdot 5^{-1/3} ) or ( 5^{-1/3}, 5^{-1/3}, 5^{2/3} )</td>
<td>( 2,1,2 ) or ( 1,1,5 )</td>
</tr>
<tr>
<td>8</td>
<td>( 9\pi^2 )</td>
<td>( 1,1,1 )</td>
<td>( 2,2,1 )</td>
</tr>
</tbody>
</table>

Let \( \lambda \in \mathbb{R}, \lambda \geq 0, \) and \( a_1,a_2,a_3 \in \mathbb{R} \) such that \( a_1a_2a_3 = 1 \) and \( a_1 \leq a_2 \leq a_3 \). With (1.4) in mind, we define

\[
E(\lambda) := \left\{ (x_1,x_2,x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq \frac{\lambda}{\pi^2} \right\}. \tag{1.5}
\]

The ellipsoid \( E(\lambda) \) has semi-axes

\[
r_1 = \frac{a_1\lambda^{1/2}}{\pi}, \ r_2 = \frac{a_2\lambda^{1/2}}{\pi}, \ r_3 = \frac{a_3\lambda^{1/2}}{\pi},
\]

and \( |E(\lambda)| = \frac{4\pi^2}{3} \lambda^{3/2} \).

By (1.4) and (1.5), we see that the Dirichlet eigenvalues \( \lambda_1(R_{a_1,a_2,a_3}), \ldots, \lambda_k(R_{a_1,a_2,a_3}) \) (counted with multiplicities) correspond to the integer lattice points that are inside or on the ellipsoid \( E(\lambda_k) \) in the first octant (excluding the coordinate planes). Thus, in order to minimise \( \lambda_k \) among all cuboids given by (1.1), we wish to determine the 3-dimensional ellipsoid \( E(\lambda) \subset \mathbb{R}^3 \) of minimal measure which encloses \( k \) integer lattice points in the first octant (excluding the coordinate planes).

For \( n \in \mathbb{N}, n \geq 2, \) estimates for the number of integer lattice points which are inside or on an \( n \)-dimensional ellipsoid have been widely studied from a number theoretical viewpoint. However, in order to use these estimates, it is crucial that the corresponding cuboids are bounded as \( k \to \infty \). As in the 2-dimensional case, this is the most difficult part of the proof.

This paper is organised as follows. In Section 2 we prove Theorem 1.1(i). In Section 3 we obtain bounds for lattice point sums which are key ingredients in the proofs of the lemmas in Section 4. In that section we follow the strategy of [2], and prove that the side-lengths of a sequence of minimal cuboids \( (R_{a_1^*,a_2^*,a_3^*})_k \) are bounded uniformly in \( k \). This is achieved by first obtaining an upper bound for the counting function \( N(\lambda) = \#\{ j \in \mathbb{N} : \lambda_j(R_{a_1,a_2,a_3}) \leq \lambda \} \) for arbitrary cuboids. Using the maximality of \( R_{a_1^*,a_2^*,a_3^*} \) and comparing with the unit cube gives the required uniform bound. Finally in Section 5 we use known estimates for the number of integer lattice points that are inside and on an ellipsoid to conclude the proof of Theorem 1.1(ii).
2 Proof of Theorem 1.1(i).

Proof. Fix $k \in \mathbb{N}$. Suppose that $\{R_{a_{i,k}^{(t)},a_{2,k}^{(t)},a_{3,k}^{(t)}}^{(t)}\}_{t \in \mathbb{N}}$ is a minimising sequence for $\lambda_k$ such that $a_{3,k}^{(t)} \to \infty$ as $t \to \infty$. In order to preserve the measure constraint $a_{1,k}^{(t)} \to 0$ as $t \to \infty$, we have that

$$
\lambda_k(R_{a_{1,k}^{(t)},a_{2,k}^{(t)},a_{3,k}^{(t)}}^{(t)}) > \frac{\pi^2}{(a_{1,k}^{(t)})^2} \to \infty, \text{ as } t \to \infty.
$$

However, for the unit cube in $\mathbb{R}^3$, $\lambda_k \leq 3\pi^2k^2 < +\infty$. This contradicts the assumption that $\{R_{a_{1,k}^{(t)},a_{2,k}^{(t)},a_{3,k}^{(t)}}^{(t)}\}_{t \in \mathbb{N}}$ is a minimising sequence for $\lambda_k$. So any minimising sequence $\{R_{a_{1,k}^{(t)},a_{2,k}^{(t)},a_{3,k}^{(t)}}^{(t)}\}_{t \in \mathbb{N}}$ for $\lambda_k$ is such that $a_{1,k}, a_{2,k}, a_{3,k}$ are bounded as $t \to \infty$. Hence, for each $i \in \{1, 2, 3\}$, there exists a convergent subsequence, again denoted by $a_{i,k}^{(t)}$ such that $a_{i,k}^{(t)} \to a_{i,k}^*$ for some $a_{i,k}^* \in (0, \infty)$. Since $(a_1, a_2, a_3) \mapsto \lambda_k(R_{a_1,a_2,a_3})$ is continuous, $\lambda_k(R_{a_{1,k}^{(t)},a_{2,k}^{(t)},a_{3,k}^{(t)}}) \to \lambda_k(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*})$ as $t \to \infty$. Hence $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ is a minimising cuboid for $\lambda_k$.

It is not difficult to see that the above argument can also be used to prove the existence of a minimising cuboid for $\lambda_k$ in $\mathbb{R}^m$ with $m \geq 4$.

3 Key lemmas to prove boundedness of an optimal cuboid.

The following lemmas are crucial in the proofs that follow in Section 4.

Lemma 3.1 Let $y \geq 0$, $a \geq 0$. For $n \in \{1, 2\}$, we have that

$$
\sum_{i=1}^{\lfloor y \rfloor/a} (y - a^2 i^2)^{n/2} \leq \frac{\sqrt{\pi}}{2a} \Gamma\left(\frac{n+1}{2}\right) y^{(n+1)/2} - \frac{1}{2} y^{n/2} + \frac{(2an)^{n/2}}{(n+2)(n+2/2)\pi} y^{n/4}.
$$

Proof. We have that

$$
\sum_{i=1}^{\lfloor y \rfloor/a} (y - a^2 i^2)^{n/2} = a^n \sum_{i=1}^{\lfloor y \rfloor/a} \left(\frac{y^{1/2}}{a} - i^2\right)^{n/2}.
$$

Let $R = \frac{y^{1/2}}{a}$ and consider $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ where

$$
g(i) = (R^2 - i^2)^{n/2}.
$$

Then, for $0 \leq i \leq R$, we have that

$$
g'(i) = -ni(R^2 - i^2)^{(n-2)/2} \leq 0,
$$

$$
g''(i) = n(R^2 - i^2)^{(n-4)/2}((n-1)i^2 - R^2) \leq 0.
$$

So $i \mapsto g(i)$ is decreasing on $[0, R]$ and, since $n = 1$ or $n = 2$, $g$ is also concave on $[0, R]$. We note that since $g$ is decreasing, $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ is the total area of the rectangles of width 1 and height $g(i)$, $i \in \{1, \ldots, \lfloor R \rfloor\}$, which are inscribed in the curve $g(x)$ for $0 \leq x \leq R$. Due to the concavity of $g$ on $(0, R)$, we can bound $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ from above by the area under $g$ minus the area of the inscribed triangles which sit on top of the aforementioned rectangles. That is

$$
\sum_{i=1}^{\lfloor R \rfloor} g(i) \leq \int_0^R g(i) \, di - \frac{1}{2} \sum_{i=1}^{\lfloor R \rfloor} (g(i-1) - g(i)) - \frac{1}{2} (R - \lfloor R \rfloor) g(\lfloor R \rfloor).
$$

(3.4)
We have that
\[
\int_0^R g(i) \, di = R^{n+1} \int_0^1 (1 - t^2)^{n/2} \, dt
\]
\[= \frac{R^{n+1}}{2} \int_0^1 (1 - s)^{n/2} \frac{1}{\sqrt{s}} \, ds = \frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{n+2}{2} \right)}{\Gamma \left( \frac{n+3}{2} \right)} R^{n+1}, \tag{3.5}
\]
where we have used [3.191.3, 8.384.1, [13]].

We also have that
\[-\frac{1}{2} \sum_{i=1}^{\lfloor R \rfloor} (g(i-1) - g(i)) - \frac{1}{2} (R - \lfloor R \rfloor) g(\lfloor R \rfloor)
\[-\frac{1}{2} R^n + \frac{1}{2} (1 + [R] - R)(R^2 - [R]^2)^{n/2}
\[-\frac{1}{2} R^n + \frac{1}{2} (1 + [R] - R)(R + [R])^{n/2}(R - [R])^{n/2}
\leq -\frac{1}{2} R^n + \frac{1}{2} (2R)^{n/2} \max_{0 \leq \beta < 1} (1 - \beta)^{3n/2}
\leq -\frac{1}{2} R^n + \frac{(2n)^{n/2}}{(n + 2)(n+2)^2} R^{n/2}. \tag{3.6}
\]
Combining (3.2), (3.4), (3.5) and (3.6) gives (3.1).

Applying the previous lemma with \(n = 1, y = a_2^2 \pi \lambda, a = a_2 \), we recover the result of Theorem 3.1 from [2]. Since \(g\) (as in (3.3)) is decreasing on \([0, \frac{a_2^2}{a_1}]\), the following holds for all \(n \in \mathbb{N}\).

Lemma 3.2 Let \(y \geq 0, a \geq 0\). For \(n \in \mathbb{N}\), we have that
\[\left\lfloor \frac{\sqrt{a / 2}}{\pi} \right\rfloor \sum_{i=1}^{\lfloor a / 2 \rfloor} (y - a^2 i^2)^{n/2} \leq \int_0^{\sqrt{a / 2}} (y - a^2 i^2)^{n/2} \, di = \frac{\sqrt{\pi}}{2a} \frac{\Gamma \left( \frac{n+2}{2} \right)}{\Gamma \left( \frac{n+3}{2} \right)} y^{(n+1)/2}.
\]

4 Uniform boundedness of an optimal cuboid.

With \(E(\lambda)\) as defined in (1.5), we define the counting function
\[N(\lambda) := \# \{ j \in \mathbb{N} : \lambda_j(R_{a_1}, a_2, a_3, a_1) \leq \lambda \} = \# \{(i_1, i_2, i_3) \in \mathbb{N}^3 \cap E(\lambda)\}.
\]
We now use the results of Section 3 to obtain an upper bound for \(N(\lambda)\).

Lemma 4.1 For \(\lambda \geq 0\) and \(a_1 \leq a_2 \leq a_3, E(\lambda), N(\lambda)\) as above, we have that
\[N(\lambda) \leq \frac{\lambda^{3/2}}{6\pi^2} - \frac{\lambda}{8\pi a_1} + \frac{\lambda^{1/2}}{16a_1^2}. \tag{4.1}
\]

Proof. For \((i_1, i_2, i_3) \in \mathbb{N}^3 \cap E(\lambda)\), we have that
\[i_3 \leq \left( \frac{a_2^2}{\pi^2} \lambda - \frac{a_2^2}{a_1^2} i_1 - \frac{a_2^2}{a_2^2} i_2 \right)^{1/2}.
\]
where “+” denotes the positive part. Hence

$$\begin{aligned} N(\lambda) &\leq \sum_{i_1 \in \mathbb{N}} \sum_{i_2 \in \mathbb{N}} \left| \left( \frac{a_3^3}{\pi^2} \lambda - \frac{a_3^2}{2a_1^2} - \frac{a_3^2}{a_2^2} \right)^{1/2} + \right| \\
&\leq \sum_{i_1=1}^{a_3^{1/2}/\pi} \sum_{i_2=1}^{a_3^{1/2}/\pi} \left( \frac{a_3^3}{\pi^2} \lambda - \frac{a_3^2}{a_1^2} - \frac{a_3^2}{a_2^2} \right)^{1/2}. \end{aligned}$$  \tag{4.2}

Applying Lemma 3.2 with \( y = \frac{a_3^2}{\pi^2} \lambda - \frac{a_3^2}{a_1^2} = \frac{a_3^2}{a_2^2}, a = \frac{a_3}{a_2}, n = 1 \) to (4.3), we have that

$$\begin{aligned} N(\lambda) &\leq \sum_{i_1=1}^{a_3^{1/2}/\pi} \frac{\pi a_2 a_3}{4a_3} \left( \frac{a_3^3}{\pi^2} \lambda - \frac{a_3^2}{2a_1^2} \right) = \sum_{i_1=1}^{a_3^{1/2}/\pi} \frac{\pi a_2 a_3}{4} \left( \frac{\lambda}{\pi^2} - \frac{\lambda}{a_1^2} \right) \\
&= \frac{\lambda^{3/2}}{6\pi^2} - \frac{\lambda}{8\pi a_1} + \frac{\lambda^{1/2}}{16a_1^2}. \end{aligned}$$  \tag{4.4}

By (4.4) and (4.5), (4.1) follows.

We now prove that the side-lengths \( a_{1,k}^*, a_{2,k}^*, a_{3,k}^* \), of an optimal cuboid \( R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*} \) in \( \mathbb{R}^3 \) are uniformly bounded.

**Lemma 4.2** For all \( k \in \mathbb{N} \),

$$a_{3,k}^* \leq 319.$$  

**Proof.** Since (4.1) holds for all \( \lambda \geq 0 \) and all cuboids, it holds for \( \lambda = \lambda_k^* \) and an optimal cuboid \( R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*} \), so

$$k \leq N(\lambda_k^*) \leq \frac{(\lambda_k^*)^{3/2}}{6\pi^2} - \frac{\lambda_k^*}{8\pi a_{1,k}^*} + \frac{(\lambda_k^*)^{1/2}}{16(a_{1,k}^*)^2},$$

and, by rearranging, we obtain that

$$\frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2 \lambda_k^*} \geq \frac{1}{8\pi a_{1,k}^*} - \frac{(\lambda_k^*)^{-1/2}}{16(a_{1,k}^*)^2}. \tag{4.6}$$

The left-hand side of (4.6) is an increasing function of \( \lambda_k^* \), so it is bounded from above by \( \frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \), where \( \nu_k \) is the \( k \)th Dirichlet eigenvalue of the Laplacian on the unit cube in \( \mathbb{R}^3 \). We obtain a lower bound for the right-hand side of (4.6) by using the fact that

$$\lambda_k^* \geq \lambda_1(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}) \geq \frac{\pi^2}{(a_{1,k}^*)^2},$$

implies that

$$-\frac{(\lambda_k^*)^{-1/2}}{16(a_{1,k}^*)^2} \geq -\frac{1}{16\pi a_{1,k}^*}. \tag{4.7}$$

Hence, by (4.7), we have that

$$\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \geq \frac{1}{16\pi a_{1,k}^*},$$
which implies that,
\[ a_{1,k}^* \geq \frac{1}{16\pi^3} \frac{6\pi^2 \nu_k}{\nu_k^{3/2} - \nu_k^{2}}. \] (4.8)

We now obtain a uniform lower bound for \( a_{1,k}^* \). Let \( \omega_3 \) denote the measure of a ball of radius 1 in \( \mathbb{R}^3 \). Then, by an estimate of Gauss, we have that
\[ N(\nu_k) = \# \left\{ (i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 \leq \frac{\nu_k}{\pi^2} \right\} \geq \frac{\omega_3}{8} \left( \frac{\nu_k^{1/2}}{\pi} - 3^{1/2} \right)^3 \geq \frac{\nu_k^{3/2}}{6\pi^2} - \frac{3^{1/2}\nu_k}{2\pi}. \]

Let \( \Theta_k \) denote the multiplicity of \( \nu_k \). Then \( N(\nu_k) \leq k + \Theta_k - 1 \). In addition, \( \Theta_k = \# \{(i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 = \frac{\nu_k}{\pi^2}\} \) is the number of integer lattice points in the first octant that lie on the sphere in \( \mathbb{R}^3 \) which is centred at \((0, 0, 0)\) and has radius \( \nu_k^{1/2} \). By projection onto the plane \( i_3 = 0 \), each of these lattice points corresponds to an integer lattice point which lies inside or on the circle \( \{(i_1, i_2) \in \mathbb{Z}^2 : i_1^2 + i_2^2 = \frac{\nu_k}{\pi}\} \) in the first quadrant. The number of integer lattice points which lie inside or on this circle is bounded from above by \( \frac{\nu_k}{2\pi} \), i.e. the area inscribed by the circle in the first quadrant. Thus we obtain that
\[ \nu_k^{3/2} \leq 6\pi^2 k + 3\pi \nu_k \left( \frac{1}{2} + 3^{1/2} \right). \] (4.9)
Hence by (4.8) and (4.9), we have that
\[ a_{1,k}^* \geq \left( 8 \left( \frac{1}{2} + 3^{1/2} \right) \right)^{-1}. \] (4.10)

Using that \( a_{1,k}^* \leq a_{2,k}^* \leq a_{3,k}^*, a_{1,k}^* a_{2,k}^* a_{3,k}^* = 1 \) and (4.10), we deduce that
\[ a_{3,k}^* \leq \frac{1}{(a_{1,k}^*)^2} \leq 64 \left( \frac{1}{2} + 3^{1/2} \right)^2 \leq 319. \]

The main obstructions to proving a corresponding result to Theorem 1.1(ii) in higher dimensions \( m \geq 4 \) are the following. Firstly, for \( m \geq 4 \) the corresponding upper bound for \( N(\lambda) \) to (4.2) involves lattice point sums \( \sum_{i=1}^{[R]} g(i) \) with \( g(i) \), \( R \) as in (3.3) and \( n \geq 3 \). For \( n \geq 3, \frac{\nu_k^{1/2}}{a\sqrt{n-1}} \) is an inflection point of \( g \) in \( (0, \frac{\nu_k^{1/2}}{a\sqrt{n-1}}) \) and so \( g \) is not concave on \( (0, \frac{\nu_k^{1/2}}{a\sqrt{n-1}}) \). Thus, the above approach cannot be used to obtain an upper bound for the left-hand side of (3.1) when \( n \geq 3 \). Secondly, the higher-dimensional equivalent of (4.1) will contain more terms in the right-hand side. The leading term in that right-hand side is the Weyl term. However, the lower order terms are bounds which are uniform in \( a_1 \), for example. Their usefulness depends on the numerical coefficients which show up. These in turn depend on lower dimensional lattice point sums.

5 Proof of Theorem 1.1(ii).

The minimisers \( R_{\lambda_{1,k}^*, a_{2,k}^*, a_{3,k}^*} \) of \( \lambda_k \) need not be unique. From this point onwards, we consider an arbitrary subsequence of minimisers denoted by \( (R_{\lambda_{1,k}^*, a_{2,k}^*, a_{3,k}^*})_k \).

For \( E(\lambda) \) as defined in (1.5), we introduce the following notation.
\[ T(\lambda) = \# \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \cap E(\lambda)\}, \]
\[ T_{x_1}(\lambda) = \# \{(0, x_2, x_3) \in \{(0) \times \mathbb{Z}^2 \cap E(\lambda)\}, \]
\[ T_{x_1}^+(\lambda) = \# \{(0, x_2, x_3) \in \{(0) \times \mathbb{N}^2 \cap E(\lambda)\}. \]
$T(\lambda)$ is the total number of integer lattice points that are inside or on the ellipsoid $E(\lambda)$ in $\mathbb{R}^3$. Similarly, $T_{x_1}(\lambda)$ is the number of integer lattice points that are inside or on the ellipse in $\mathbb{R}^2$ which is centred at $(0, 0)$ and has semi-axes $\frac{a\lambda^{1/2}}{\pi}, \frac{a\lambda^{1/2}}{\pi}$. $T^+_x(\lambda)$ is the number of these lattice points that lie in the first quadrant (excluding the axes). $T_{x_1}(\lambda), T_{x_2}^+(\lambda)$ etc. are defined similarly. Thus, we have that

$$T(\lambda) = 8N(\lambda) + 4T_{x_1}^+(\lambda) + 4T_{x_2}^+(\lambda) + 4T_{x_3}^+(\lambda) + 2\left\lceil \frac{a_1\lambda^{1/2}}{\pi} \right\rceil + 2\left\lceil \frac{a_2\lambda^{1/2}}{\pi} \right\rceil + 2\left\lceil \frac{a_3\lambda^{1/2}}{\pi} \right\rceil + 1,$$

which implies that

$$N(\lambda) = \frac{1}{8} T(\lambda) - \frac{1}{2} T_{x_1}^+(\lambda) - \frac{1}{2} T_{x_2}^+(\lambda) - \frac{1}{2} T_{x_3}^+(\lambda) - \frac{1}{4} a_1\lambda^{1/2} - \frac{1}{4} a_2\lambda^{1/2} - \frac{1}{4} a_3\lambda^{1/2} - \frac{1}{8}.$$

In addition, we have that

$$T_{x_1}(\lambda) = 4T_{x_1}^+(\lambda) + 2\left\lceil \frac{a_2\lambda^{1/2}}{\pi} \right\rceil + 2\left\lceil \frac{a_3\lambda^{1/2}}{\pi} \right\rceil + 1,$$

which implies that

$$T_{x_1}^+(\lambda) = \frac{1}{4} T_{x_1}(\lambda) - \frac{1}{2} a_2\lambda^{1/2} - \frac{1}{2} a_3\lambda^{1/2} - \frac{1}{4},$$

and similarly for $T_{x_2}^+(\lambda), T_{x_3}^+(\lambda)$. Thus, we obtain

$$N(\lambda) = \frac{1}{8} T(\lambda) - \frac{1}{8} T_{x_1}(\lambda) - \frac{1}{8} T_{x_2}(\lambda) - \frac{1}{8} T_{x_3}(\lambda) + \frac{1}{8} a_1\lambda^{1/2} + \frac{1}{4}\left\lceil \frac{a_2\lambda^{1/2}}{\pi} \right\rceil + \frac{1}{4}\left\lceil \frac{a_3\lambda^{1/2}}{\pi} \right\rceil + \frac{1}{4}. \quad (5.1)$$

Below we use this expression for $N(\lambda)$ in order to prove Theorem 1.1(ii).

**Proof of Theorem 1.1(ii).** By setting $\lambda = \lambda_k^*$ in (5.1) and considering an optimal cuboid $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$, we have that

$$k \leq N(\lambda_k^*) = \frac{1}{8} T(\lambda_k^*) - \frac{1}{8} T_{x_1}(\lambda_k^*) - \frac{1}{8} T_{x_2}(\lambda_k^*) - \frac{1}{8} T_{x_3}(\lambda_k^*) + \frac{1}{8} a_1^*\lambda_k^{1/2} + \frac{1}{4}\left\lceil \frac{a_2^*\lambda_k^{1/2}}{\pi} \right\rceil + \frac{1}{4}\left\lceil \frac{a_3^*\lambda_k^{1/2}}{\pi} \right\rceil + \frac{1}{4}. \quad (5.2)$$

By Lemma 4.2, the $\{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*\}$ are uniformly bounded, so it is possible to make use of known estimates for the number of integer lattice points that are inside or on a 3-dimensional ellipsoid or a 2-dimensional ellipse. In particular there exists $C < \infty$ such that for all $\lambda \geq 0$

$$\frac{4}{3\pi^2}\lambda^{3/2} - C\lambda^{3/2} \leq T(\lambda) \leq \frac{4}{3\pi^2}\lambda^{3/2} + C\lambda^{3/2} + 1, \quad (5.3)$$

where $\beta$ is as defined in the Introduction. Similarly there exists $D < \infty$ such that for all $\lambda \geq 0$

$$\frac{4a_2a_3}{\pi}\lambda - D\lambda^{3/2} \leq T_{x_1}(\lambda) \leq \frac{4a_2a_3}{\pi}\lambda + D\lambda^{3/2} + 1, \quad (5.4)$$

where $\theta$ is an exponent of the remainder in Gauss’ circle problem

$$\# \{(i_1,i_2) \in \mathbb{Z}^2 : i_1^2 + i_2^2 \leq R^2\} - \pi R^2 = O(R^\theta), R \to \infty.$$
The best known estimate to date is $\theta > \frac{131}{200}$, see the Introduction in [16]. Hence the formula above holds for $\theta = \frac{131}{200} + \epsilon$ for any $\epsilon > 0$. The corresponding inequalities to (5.4) also hold for $T_{x_2}(\lambda), T_{x_3}(\lambda)$. Using these inequalities and (5.2), we obtain the following upper bound for $N(\lambda_k^*)$.

$$k \leq N(\lambda_k^*) \leq \frac{(\lambda_k^*)^{3/2}}{6\pi^2} - \frac{1}{8\pi} \left( \frac{1}{a_{1,k}} + \frac{1}{a_{2,k}} + \frac{1}{a_{3,k}} \right) \lambda_k^* + \frac{C}{8}(\lambda_k^*)^{3/2}$$

$$+ \frac{1}{4\pi} (a_{1,k}^* + a_{2,k}^* + a_{3,k}^*)(\lambda_k^*)^{1/2} + \frac{3D}{8}(\lambda_k^*)^{\theta/2} + \frac{3}{8}.$$

(5.5)

Rearranging (5.5), we obtain that

$$\frac{1}{a_{1,k}} + \frac{1}{a_{2,k}} + \frac{1}{a_{3,k}} \leq 8\pi \left( \frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \right) + \pi C(\lambda_k^*)^{-(2-\beta)/2} + 2(a_{1,k}^* + a_{2,k}^* + a_{3,k}^*)(\lambda_k^*)^{-1/2} + 3\pi D(\lambda_k^*)^{-(2-\theta)/2} + 3\pi(\lambda_k^*)^{-1}.$$}

Since $\frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k}$ is an increasing function of $\lambda_k^*$, we can replace $\lambda_k^*$ by $\nu_k$, where $\nu_k$ is the $k$th Dirichlet eigenvalue of the Laplacian on the unit cube in $\mathbb{R}^3$. Thus, by Pólya’s Inequality $\lambda_k^* \geq (6\pi^2 k)^{2/3}$, ([19, 20]), we obtain

$$\frac{1}{a_{1,k}} + \frac{1}{a_{2,k}} + \frac{1}{a_{3,k}} \leq 8\pi \left( \frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \right) + \pi C(\nu_k)^{-(2-\beta)/2} + 2(\nu_k^* + a_{2,k}^* + a_{3,k}^*)(\lambda_k^*)^{-1/2} + 3\pi D(\lambda_k^*)^{-(2-\theta)/2} + 3\pi(\lambda_k^*)^{-1}.$$}

(5.6)

To obtain an upper bound for $\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k}$ we proceed as follows. By (5.1) with $\lambda = \nu_k$ we have that

$$N(\nu_k) = \frac{1}{8} T(\nu_k) - \frac{3}{8} T_{x_1}(\nu_k) + \frac{3}{4} \left[ \frac{\nu_k^{1/2}}{\pi} \right] + \frac{1}{4}.$$}

(5.7)

Since $a_1 = a_2 = a_3 = 1$, by (5.3) and (5.4), we have that

$$\frac{4}{3\pi^2} \nu_k^{3/2} - C \nu_k^{\beta/2} \leq T(\nu_k),$$}

(5.8)

and

$$T_{x_1}(\nu_k) \leq \frac{\nu_k}{\pi} + D \nu_k^{\theta/2} + 1,$$

(5.9)

where $\beta$ and $\theta$ are as in (5.3), (5.4). Again let $\Theta_k$ denote the multiplicity of $\nu_k$. Thus by (5.7), (5.8) and (5.9), we obtain a lower bound for $N(\nu_k)$:

$$k + \Theta_k - 1 \geq N(\nu_k) \geq \frac{\nu_k^{3/2}}{6\pi^2} - \frac{C}{8} \nu_k^{\beta/2} - \frac{3}{8\pi} \nu_k - \frac{3D}{8} \nu_k^{\theta/2} + \frac{3}{4\pi} \nu_k^{1/2} - \frac{7}{8},$$}

which implies that

$$\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \leq \frac{3}{8\pi} + \frac{C}{8} \nu_k^{-(2-\beta)/2} + \frac{3D}{8} \nu_k^{-(2-\theta)/2} - \frac{3}{4\pi} \nu_k^{1/2} - \Theta_k \nu_k^{1/2} - \frac{3}{8} \nu_k^{1/2} - \frac{7}{8},$$}

$$\leq \frac{3}{8\pi} + \frac{C}{8} \nu_k^{-(2-\beta)/2} + \frac{3D}{8} \nu_k^{-(2-\theta)/2} + \Theta_k \nu_k^{1/2} + \frac{3}{8} \nu_k^{-(2-\beta)/2} - \frac{3}{8} \nu_k^{-(2-\theta)/2} + \Theta_k \nu_k^{1/2}.$$

9
by Pólya’s Inequality.

We have that $\Theta_k = \# \{(i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 = \frac{k^3}{2} \}$ is the number of integer lattice points in the first octant that lie on the sphere in $\mathbb{R}^3$ which is centred at $(0, 0, 0)$ and has radius $\frac{k^{3/2}}{\pi}$. It is well known that $\# \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d\} = O(d^{1/2} + o(1))$.

The following routine proof was communicated by T. Wooley. Let $n = d - x_3^2$. Now $|x_3| \leq d^{1/2}$, so for $x_3 \in [-d^{1/2}, d^{1/2}] \cap \mathbb{Z}$, there are at most $2d^{1/2} + 1$ possible values of $n$. If $n = 0$, then $x_1^2 + x_2^2 = 0$ has one solution $(0, 0) \in \mathbb{Z}^2$. Suppose that $n \neq 0$. Let $R(n)$ denote the number of pairs $(x_1, x_2) \in \mathbb{Z}^2$ such that $x_1^2 + x_2^2 = n$. Then

$$\# \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d\} \leq 1 + \sum_{|z| \leq d^{1/2}} R(d - z^2).$$

By Corollary 3.23 of [18], we have that

$$R(n) = 4 \sum_{d|n, d > 0, d \text{ odd}} \left( -\frac{1}{d} \right),$$

where the sum is taken over all positive, odd divisors of $n$ and $\left( -\frac{1}{d} \right)$ is the quadratic residue symbol. Thus $R(n) \leq 4D(n)$, where $D(n)$ denotes the number of positive divisors of $n$. By Theorem 8.31 of [18], for every $\epsilon > 0$, there exists $n_\epsilon$ such that for $n > n_\epsilon$,

$$D(n) \leq n^{(1+\epsilon) \log 2 / \log \log n},$$

which implies that $D(n) = O(n^\epsilon)$. Therefore we obtain that

$$\# \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d\} \leq 1 + O \left( \sum_{|z| \leq d^{1/2}} (d - z^2)^\epsilon \right) \leq 1 + O(d^{1/2 + \epsilon}).$$

So $\Theta_k = O(\nu_k^{1/2 + o(1)})$ and $\Theta_k \nu_k^{-1} = O(\nu_k^{-1/2 + o(1)}) = O(k^{-1/2 + o(1)})$. Thus we obtain

$$\frac{\nu_k^{3/2} - 6\pi^2k}{6\pi^2\nu_k} \leq \frac{3}{8\pi} + O(k^{-2-\beta/3}). \quad (5.10)$$

So by (5.6) and (5.10), we deduce that

$$\frac{1}{a_{1,k}} + \frac{1}{a_{2,k}} + \frac{1}{a_{3,k}} \leq 3 + O(k^{-2-\beta/3}), k \to \infty. \quad (5.11)$$

Furthermore, by the Arithmetic Mean – Geometric Mean Inequality applied to $\frac{1}{a_{1,k}} + \frac{1}{a_{2,k}}$, we have by (5.11) that

$$2(a_{3,k}^*)^{1/2} + \frac{1}{a_{3,k}^*} \leq 3 + O(k^{-2-\beta/3}), k \to \infty.$$ 

Let $a_{3,k}^* = 1 + \delta_k$ where $\delta_k > 0$. Then

$$2(1 + \delta_k)^{3/2} + 1 \leq 3 + 3\delta_k + O(k^{-2-\beta/3}), k \to \infty.$$ 

Since $a_{3,k}^* \leq 319$, $\delta_k \leq 399$. Hence $(1 + \delta_k)^{3/2} \geq 1 + \frac{3}{2}\delta_k + \frac{3}{1600}\delta_k^2$ for $0 \leq \delta_k \leq 399$, we deduce that $\delta_k \leq O(k^{-2-\beta/6}), k \to \infty$. As this estimate is independent of the subsequence $(R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*})_k$, we arrive at the conclusion of Theorem 1.1(ii).
References


