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Minimising Dirichlet eigenvalues on cuboids of unit measure

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Abstract

We consider the minimisation of Dirichlet eigenvalues \( \lambda_k \), \( k \in \mathbb{N} \), of the Laplacian on cuboids of unit measure in \( \mathbb{R}^3 \). We prove that any sequence of optimal cuboids in \( \mathbb{R}^3 \) converges to a cube of unit measure in the sense of Hausdorff as \( k \to \infty \). We also obtain an upper bound for that rate of convergence.

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1 Introduction.

The eigenvalues of the Laplacian have been the object of intensive study over the last century. Of particular interest are related shape optimisation problems. For \( k \in \mathbb{N} \), the goal is to optimise the \( k \)’th eigenvalue of the Laplacian with boundary conditions over a collection of open sets in \( \mathbb{R}^m \). This collection satisfies geometric constraints, such as fixed Lebesgue measure or fixed perimeter.

For an open set \( \Omega \subset \mathbb{R}^m \), \( m \geq 2 \), of finite Lebesgue measure \( |\Omega| \), we let \( \lambda_k(\Omega) \), \( k \in \mathbb{N} \), denote the Dirichlet eigenvalues of the Laplacian on \( \Omega \) which are strictly positive, arranged in non-decreasing order and counted with multiplicity:

\[
\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \cdots
\]

This sequence accumulates at \(+\infty\).

We consider the following minimisation problem:

\[
\lambda^*_k(m) := \inf \{ \lambda_k(\Omega) : \Omega \text{ open in } \mathbb{R}^m, |\Omega| = c \}.
\]

It was shown by Faber and Krahn that among all open sets in \( \mathbb{R}^m \) of measure \( c \), the ball of measure \( c \) minimises the first Dirichlet eigenvalue, see [15]. Krahn and Szegö proved that, among all open sets in \( \mathbb{R}^m \) of measure \( c \), the second Dirichlet eigenvalue is minimised by the union of two disjoint balls of measure \( \frac{c}{2} \) each, see [15]. For \( k \geq 3 \), the existence of an open set of prescribed measure which minimises the \( k \)’th Dirichlet eigenvalue remains unresolved to date. However, in the class of quasi-open sets of prescribed measure, it was shown by Bucur in [7] that a minimiser does exist and that such a minimiser is bounded and has finite perimeter. Independently, Mazzoleni and Pratelli proved the existence of a minimiser in [17] in the collection of quasi-open sets. For any lower semi-continuous,
increasing function of the first \( k \) Dirichlet eigenvalues, they proved the existence of a minimiser which is bounded in terms of \( k \) and \( m \) independently of the function. It was shown in [5] that for \( k \leq m+1 \), any bounded minimiser of \( \lambda_k(\Omega) \) has at most \( \min\{7, k\} \) components.

No optimal domains are known for \( \lambda_k \) with \( k \geq 3 \). In particular, the conjecture that if \( m = 2 \), then \( \lambda_3(\Omega) \) is bounded from below by the third eigenvalue of the disc with the same measure as \( \Omega \) is open. There are no obvious candidates for minimisers of \( \lambda_k \) with \( k \geq 5 \) in any dimension \( m \geq 2 \). Even for \( m = 2 \), minimisers need not be discs or disjoint unions of discs, see [21]. Furthermore, it was shown in [6] that for \( k \geq 5 \), \( \lambda_k(\Omega) \) cannot be minimised by a disc or a disjoint union of discs. The numerical investigation [1] suggests that for some values of \( k \) the minimisers may not have any symmetries.

Pólya’s conjecture for Dirichlet eigenvalues asserts that for all bounded, open sets \( \Omega \subset \mathbb{R}^m \), \( \lambda_k(\Omega) \geq 4\pi^2(\omega_m(\Omega))^{-2/m}k^{2/m} \), where \( \omega_m \) denotes the measure of a ball in \( \mathbb{R}^m \) of radius 1. It was shown in [11] that Pólya’s conjecture is equivalent to \( \lambda_k^*(m) \) being asymptotically equal to \( 4\pi^2(\omega_m c)^{-2/m}k^{2/m} \) as \( k \to \infty \).

It is also interesting to consider the optimisation of the eigenvalues of the Laplacian subject to other geometric constraints, such as fixed perimeter. For the Dirichlet eigenvalues, existence of a minimiser in the class of open sets in \( \mathbb{R}^m \) of finite Lebesgue measure and prescribed perimeter was shown in [12]. Moreover, it was shown there that any minimiser is bounded and connected, and regularity results for the boundary were also obtained. Bucur and Freitas, [9], showed that any sequence of minimisers of \( \lambda_k \) in \( \mathbb{R}^2 \) with perimeter \( \ell \) converges in the sense of Hausdorff to the disc of perimeter \( \ell \) as \( k \to \infty \). They also showed that if the collection of admissible sets is restricted to the collection of \( n \)-sided, convex, planar polygons of perimeter \( \ell \), then any sequence of minimisers converges to the regular \( n \)-sided polygon of perimeter \( \ell \) as \( k \to \infty \). For \( m \geq 2 \), other constraints were considered in [4], including perimeter and moment of inertia, subject to an additional convexity constraint. Further results for the Dirichlet eigenvalues were obtained in [3], [8], [5], [9] and [4]. Some of the results of [3] follow directly from those in [4], while the results of [12] supersede those of [8].

Recently, Antunes and Freitas considered the problem of minimising \( \lambda_k \) over all planar rectangles of unit measure, [2]. In Theorem 2.1 of [2], they showed that any sequence of minimising rectangles for the Dirichlet eigenvalues converges to the unit square in the sense of Hausdorff as \( k \to \infty \).

In Theorem 1.1 below we obtain the corresponding 3-dimensional result for the Dirichlet eigenvalues of the Laplacian on cuboids in \( \mathbb{R}^3 \) of unit measure. In addition we obtain an estimate for the rate of convergence. Let \( R_{a_1,a_2,a_3} \) denote a cuboid in \( \mathbb{R}^3 \) of side-lengths \( a_1, a_2, a_3 \) such that \( a_1a_2a_3 = 1 \) and \( a_1 \leq a_2 \leq a_3 \),

\[
R_{a_1,a_2,a_3} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 0 < x_1 < a_1, 0 < x_2 < a_2, 0 < x_3 < (a_1a_2)^{-1}, a_1 \leq a_2 \leq a_3\}. \tag{1.1}
\]

We prove the following.

**Theorem 1.1**  
(i) Let \( k \in \mathbb{N} \). The variational problem

\[
\lambda_k^* := \inf \{ \lambda_k(R_{a_1,a_2,a_3}) : a_1 \leq a_2 \leq a_3 \}
\]

has a minimising cuboid \( R_{a_{1,1},a_{2,1},a_{3,1}} \) with side-lengths

\[
a_{1,1}^* \leq a_{2,1}^* \leq a_{3,1}^*, \text{ such that } a_{1,1}^*a_{2,1}^*a_{3,1}^* = 1.
\]

(ii)

\[
a_{3,k}^* \leq 1 + O(k^{-(2-\beta)/6}), \quad k \to \infty, \tag{1.2}
\]

where \( \beta \) is an exponent of the remainder in

\[
\#\{(i_1,i_2,i_3) \in \mathbb{Z}^3 : i_1^2 + i_2^2 + i_3^2 \leq R^2\} - \frac{4\pi}{3}R^3 = O(R^3), \quad R \to \infty.
\]

Furthermore, any sequence of optimal cuboids \( R_{a_{1,1,k}^*,a_{2,1,k}^*,a_{3,1,k}^*} \) converges to the unit cube in \( \mathbb{R}^3 \) in the sense of Hausdorff as \( k \to \infty \).
The best known estimate to date is that for any $\epsilon > 0$, $\beta = \frac{21}{10} + \epsilon$, see [14]. Hence (1.2) holds for $\beta = \frac{21}{10} + \epsilon, \epsilon > 0$. The conjecture for the optimal remainder is $\beta = 1 + \epsilon, \epsilon > 0$. See [10].

A heuristic explanation for this asymptotic shape result is the following (see also [2]). For any cuboid $R$ in $\mathbb{R}^3$ with measure $|R|$ and perimeter $\text{Per}(R)$, one has that

$$
\lambda_k(R) = \left(\frac{6\pi^2 k}{|R|}\right)^{2/3} + \frac{(3\pi^5)^{1/3}\text{Per}(R)k^{1/3}}{2^{5/3}|R|^{1/3}} + o(k^{1/3}), \ k \to \infty.
$$

(1.3)

So if $|R| = 1$ then (1.3) suggests that the cuboid that minimises $\lambda_k(R)$, $k \to \infty$, is the one with minimal perimeter, i.e. the unit cube.

The Dirichlet eigenvalues of the Laplacian on a cuboid $R_{a_1,a_2,a_3}$ (as in (1.1)) are given by

$$
\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2}, \ i_1, i_2, i_3 \in \mathbb{N}.
$$

(1.4)

By listing these in non-decreasing order including multiplicities, the $k$'th Dirichlet eigenvalue on $R_{a_1,a_2,a_3}$, $\lambda_k(R_{a_1,a_2,a_3})$, is the $k$'th item of this list. In the table below we list the minimising cuboids for the first few Dirichlet eigenvalues.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda_k$</th>
<th>$a_{1,k}, a_{2,k}, a_{3,k}$</th>
<th>Minimising modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3\pi^2$</td>
<td>1, 1, 1</td>
<td>(1, 1, 1)</td>
</tr>
<tr>
<td>2</td>
<td>$3 \cdot 2^{2/3} \pi^2$</td>
<td>$2^{-1/3}, 2^{-1/3}, 2^{2/3}$</td>
<td>(1, 1, 2)</td>
</tr>
<tr>
<td>3</td>
<td>$3 \cdot 2^{-2/3} 3^{2/3} \pi^2$</td>
<td>$(\frac{1}{2})^{1/3}, (\frac{3}{2})^{1/3}, (\frac{3}{2})^{1/3}$</td>
<td>(1, 2, 1)</td>
</tr>
<tr>
<td>4</td>
<td>$6\pi^2$</td>
<td>1, 1, 1</td>
<td>(1, 1, 1)</td>
</tr>
<tr>
<td>5</td>
<td>$3^{2/3} \pi^2$</td>
<td>$3^{-1/3}, 3^{-1/3}, 3^{2/3}$</td>
<td>(1, 1, 3)</td>
</tr>
<tr>
<td>6</td>
<td>$3 \cdot 2^{4/3} 3^{2/3} \pi^2$</td>
<td>$2^{-2/3}, 2^{1/3}, 2^{1/3}$ or $2^{-2/3}, 2^{-2/3}, 2^{4/3}$</td>
<td>(1, 2, 2) or (1, 1, 4)</td>
</tr>
<tr>
<td>7</td>
<td>$3 \cdot 2^{-3/3} 3^{2/3} \pi^2$</td>
<td>$5^{1/3}, 5^{1/3}, 5^{-1/3}$ or $5^{-1/3}, 5^{-1/3}, 5^{2/3}$</td>
<td>(2, 1, 2) or (1, 1, 5)</td>
</tr>
<tr>
<td>8</td>
<td>$9\pi^2$</td>
<td>1, 1, 1</td>
<td>(2, 2, 1)</td>
</tr>
</tbody>
</table>

Let $\lambda \in \mathbb{R}$, $\lambda \geq 0$, and $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 a_2 a_3 = 1$ and $a_1 \leq a_2 \leq a_3$. With (1.4) in mind, we define

$$
E(\lambda) := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq \frac{\lambda}{\pi^2} \right\}.
$$

(1.5)

The ellipsoid $E(\lambda)$ has semi-axes

$$
    r_1 = \frac{a_1 \lambda^{1/2}}{\pi}, \quad r_2 = \frac{a_2 \lambda^{1/2}}{\pi}, \quad r_3 = \frac{a_3 \lambda^{1/2}}{\pi},
$$

and $|E(\lambda)| = \frac{4}{3\pi^2} \lambda^{3/2}$.

By (1.4) and (1.5), we see that the Dirichlet eigenvalues $\lambda_1(R_{a_1,a_2,a_3}), \ldots, \lambda_k(R_{a_1,a_2,a_3})$ (counted with multiplicities) correspond to the integer lattice points that are inside or on the ellipsoid $E(\lambda_k)$ in the first octant (excluding the coordinate planes). Thus, in order to minimise $\lambda_k$ among all cuboids given by (1.1), we wish to determine the 3-dimensional ellipsoid $E(\lambda) \subset \mathbb{R}^3$ of minimal measure which encloses $k$ integer lattice points in the first octant (excluding the coordinate planes).

For $n \in \mathbb{N}$, $n \geq 2$, estimates for the number of integer lattice points which are inside or on an $n$-dimensional ellipsoid have been widely studied from a number theoretical viewpoint. However, in order to use these estimates, it is crucial that the corresponding cuboids are bounded as $k \to \infty$. As in the 2-dimensional case, this is the most difficult part of the proof.

This paper is organised as follows. In Section 2 we prove Theorem 1.1(i). In Section 3 we obtain bounds for lattice point sums which are key ingredients in the proofs of the lemmas in Section 4. In that section we follow the strategy of [2], and prove that the side-lengths of a sequence of minimal cuboids $(R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*})_k$ are bounded uniformly in $k$. This is achieved by first obtaining an upper bound for the counting function $N(\lambda) = \# \{ j \in \mathbb{N} : \lambda_j(R_{a_1,a_2,a_3}) \leq \lambda \}$ for arbitrary cuboids. Using the maximality of $R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*}$, and comparing with the unit cube gives the required uniform bound. Finally in Section 5 we use known estimates for the number of integer lattice points that are inside and on an ellipsoid to conclude the proof of Theorem 1.1(ii).
2 Proof of Theorem 1.1(i).

Proof. Fix $k \in \mathbb{N}$. Suppose that $\{R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}^{(\ell)}\}_{\ell \in \mathbb{N}}$ is a minimising sequence for $\lambda_k$ such that $a_{3,k}^{(\ell)} \to \infty$ as $\ell \to \infty$. In order to preserve the measure constraint $a_{3,k}^{(\ell)} \to 0$ as $\ell \to \infty$. So, we have that

$$\lambda_k(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}^{(\ell)}) > \frac{\pi^2}{(a_{3,k}^{(\ell)})^2} \to \infty, \text{ as } \ell \to \infty.$$ 

However, for the unit cube in $\mathbb{R}^3$, $\lambda_k \leq 3\pi^2 k^2 < +\infty$. This contradicts the assumption that $\{R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}^{(\ell)}\}_{\ell \in \mathbb{N}}$ is a minimising sequence for $\lambda_k$. So any minimising sequence $\{R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}^{(\ell)}\}_{\ell \in \mathbb{N}}$ for $\lambda_k$ is such that $a_{1,k}, a_{2,k}, a_{3,k}$ are bounded as $\ell \to \infty$. Hence, for each $i \in \{1, 2, 3\}$, there exists a convergent subsequence, again denoted by $a_{i,k}^{(\ell)}$ such that $a_{i,k}^{(\ell)} \to a_{i,k}^*$ for some $a_{i,k}^* \in (0, \infty)$. Since $(a_1, a_2, a_3) \mapsto \lambda_k(R_{a_1,a_2,a_3})$ is continuous, $\lambda_k(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}^{(\ell)}) \to \lambda_k(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}^{*})$ as $\ell \to \infty$. Hence $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}^*$ is a minimising cuboid for $\lambda_k$.

It is not difficult to see that the above argument can also be used to prove the existence of a minimising cuboid for $\lambda_k$ in $\mathbb{R}^m$ with $m \geq 4$.

3 Key lemmas to prove boundedness of an optimal cuboid.

The following lemmas are crucial in the proofs that follow in Section 4.

Lemma 3.1 Let $y \geq 0$, $a \geq 0$. For $n \in \{1, 2\}$, we have that

$$\sum_{i=1}^{\left\lfloor \frac{y}{a} \right\rfloor} (y - a^2 i^2)^{n/2} \leq \sqrt{\pi} \frac{\Gamma \left( \frac{n+2}{2} \right)}{2a \Gamma \left( \frac{n+1}{2} \right)} y^{(n+1)/2} - \frac{1}{2} y^{n/2} + \frac{(2a)^{n/2}}{(n+2)(n+2)/2} y^{n/4}. \tag{3.1}$$

Proof. We have that

$$\sum_{i=1}^{\left\lfloor \frac{y}{a} \right\rfloor} (y - a^2 i^2)^{n/2} = a^n \sum_{i=1}^{\left\lfloor \frac{y}{a} \right\rfloor} \left( \frac{y^{1/2}}{a} \right)^2 - i^2 \right)^{n/2}. \tag{3.2}$$

Let $R = \frac{y^{1/2}}{a}$ and consider $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ where

$$g(i) = (R^2 - i^2)^{n/2}. \tag{3.3}$$

Then, for $0 \leq i \leq R$, we have that

$$g'(i) = -ni(R^2 - i^2)^{(n-2)/2} \leq 0,$$

$$g''(i) = n(R^2 - i^2)^{(n-4)/2}((n-1)i^2 - R^2) \leq 0.$$ 

So $i \mapsto g(i)$ is decreasing on $[0, R]$ and, since $n = 1$ or $n = 2$, $g$ is also concave on $[0, R]$. We note that since $g$ is decreasing, $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ is the total area of the rectangles of width 1 and height $g(i)$, $i \in \{1, \ldots, \lfloor R \rfloor\}$, which are inscribed in the curve $g(x)$ for $0 \leq x \leq R$. Due to the concavity of $g$ on $(0, R)$, we can bound $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ from above by the area under $g$ minus the area of the inscribed triangles which sit on top of the aforementioned rectangles. That is

$$\sum_{i=1}^{\lfloor R \rfloor} g(i) \leq \int_0^R g(i) \, di - \frac{1}{2} \sum_{i=1}^{\lfloor R \rfloor} (g(i-1) - g(i)) - \frac{1}{2} \left( R - \lfloor R \rfloor \right) g(\lfloor R \rfloor). \tag{3.4}$$
We have that
\[
\int_0^R g(i) \, di = R^{n+1} \int_0^1 (1 - t^2)^{n/2} \, dt = \frac{R^{n+1}}{2} \int_0^1 (1 - s)^{n/2} \frac{1}{\sqrt{s}} \, ds = \frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{n+2}{2} \right)}{\Gamma \left( \frac{n+3}{2} \right)} R^{n+1},
\]
(3.5)
where we have used [3.191.3, 8.384.1, [13]].

We also have that
\[
- \frac{1}{2} \sum_{i=1}^{\lfloor R \rfloor} (g(i-1) - g(i)) - \frac{1}{2} (R - \lfloor R \rfloor) g(\lfloor R \rfloor)
\]
\[
= - \frac{1}{2} R^n + \frac{1}{2} (1 + \lfloor R \rfloor - R)(R^2 - \lfloor R \rfloor^2)^{n/2}
\]
\[
= - \frac{1}{2} R^n + \frac{1}{2} (1 + \lfloor R \rfloor - R)(R + \lfloor R \rfloor)^{n/2}(R - \lfloor R \rfloor)^{n/2}
\]
\[
\leq - \frac{1}{2} R^n + \frac{1}{2} (2R)^{n/2} \max_{0 \leq \beta < 1} (1 - \beta)^{3n/2}
\]
\[
= - \frac{1}{2} R^n + \frac{(2n)^{n/2}}{n + 2(n+2)^{n/2}} R^{n/2}.
\]
(3.6)
Combining (3.2), (3.4), (3.5) and (3.6) gives (3.1).

Applying the previous lemma with \( n = 1, y = \frac{a^2}{\pi^2} \lambda, \) and \( a = \frac{a_2}{a_1}, \) we recover the result of Theorem 3.1 from [2]. Since \( g \) (as in (3.3)) is decreasing on \([0, \frac{1}{2}a_2],\) the following holds for all \( n \in \mathbb{N}.\)

**Lemma 3.2** Let \( y \geq 0, a \geq 0. \) For \( n \in \mathbb{N}, \) we have that
\[
\left\lfloor \frac{1}{2} \right\lfloor y^{1/2} \right\rfloor \sum_{i=1}^{\lfloor y^{1/2} \rfloor} (y - a^2 i^2)^{n/2} \leq \int_0^{\lfloor y^{1/2} \rfloor} (y - a^2 i^2)^{n/2} \, di = \frac{\sqrt{\pi}}{2a} \frac{\Gamma \left( \frac{n+2}{2} \right)}{\Gamma \left( \frac{n+3}{2} \right)} y^{(n+1)/2}.
\]

### 4 Uniform boundedness of an optimal cuboid.

With \( E(\lambda) \) as defined in (1.5), we define the counting function
\[
N(\lambda) := \# \{ j \in \mathbb{N} : \lambda_j (R_{a_1,a_2,a_3}) \leq \lambda \} = \# \{(i_1, i_2, i_3) \in \mathbb{N}^3 \cap E(\lambda)\}.
\]
We now use the results of Section 3 to obtain an upper bound for \( N(\lambda).\)

**Lemma 4.1** For \( \lambda \geq 0 \) and \( a_1 \leq a_2 \leq a_3, \) \( E(\lambda), \) \( N(\lambda) \) as above, we have that
\[
N(\lambda) \leq \frac{\lambda^{3/2}}{6\pi^2} - \frac{\lambda}{8\pi a_1} + \frac{\lambda^{1/2}}{16a_1^2}.
\]
(4.1)

**Proof.** For \((i_1, i_2, i_3) \in \mathbb{N}^3 \cap E(\lambda),\) we have that
\[
i_3 \leq \left\lfloor \left( \frac{a_2^2}{\pi^2} \lambda - \frac{a_1^2}{a_1^2} i_1 - \frac{a_3^2}{a_2^2} i_2 \right)^{1/2} \right\rfloor.
\]
where “+” denotes the positive part. Hence

\[ N(\lambda) \leq \sum_{i_1 \in N} \sum_{i_2 \in N} \left[ \left( \frac{a_1}{\pi^2} \lambda - \frac{a_1}{a_1^i} - \frac{a_2}{a_2^i} \right)^{1/2} \right] \]  

(4.2)

\[ \leq \sum_{i_1=1}^{a_1^{1/2}} \sum_{i_2=1}^{\lambda \pi^2} \left( \frac{a_1}{\pi^2} \lambda - \frac{a_1}{a_1^i} - \frac{a_2}{a_2^i} \right)^{1/2}. \]  

(4.3)

Applying Lemma 3.2 with \( y = \frac{a_1}{\pi^2} \lambda - \frac{a_1}{a_1^i}, n = 1 \) to (4.3), we have that

\[ N(\lambda) \leq \sum_{i_1=1}^{a_1^{1/2}} \frac{\pi a_2 a_3}{4 a_3} \left( \frac{\lambda}{\pi^2} - \frac{i_1^2}{a_1^i} \right) = \sum_{i_1=1}^{a_1^{1/2}} \frac{\pi a_2 a_3}{4} \left( \frac{\lambda}{\pi^2} - \frac{i_1^2}{a_1^i} \right). \]  

(4.4)

Applying Lemma 3.1 with \( y = \frac{\lambda}{\pi^2}, n = 2 \), we obtain that

\[ \frac{\pi a_2 a_3}{4} \sum_{i_1=1}^{a_1^{1/2}} \left( \frac{\lambda}{\pi^2} - \frac{i_1^2}{a_1^i} \right) \leq \frac{\pi a_2 a_3}{4} \left( \frac{2a_1}{3\pi^3} \lambda^{3/2} - \frac{1}{2\pi^2} \lambda + \frac{1}{4\pi a_1} \lambda^{1/2} \right) \]

\[ = \frac{\lambda^{3/2}}{6\pi^2} - \frac{\lambda}{8\pi a_1} + \frac{\lambda^{1/2}}{16a_1^2}. \]  

(4.5)

By (4.4) and (4.5), (4.1) follows. \[ \blacksquare \]

We now prove that the side-lengths \( a_{1,k}, a_{2,k}, a_{3,k} \), of an optimal cuboid \( R_{a_{1,k},a_{2,k},a_{3,k}} \) in \( \mathbb{R}^3 \) are uniformly bounded.

**Lemma 4.2** For all \( k \in \mathbb{N} \),

\[ a_{3,k} \leq 319. \]

**Proof.** Since (4.1) holds for all \( \lambda \geq 0 \) and all cuboids, it holds for \( \lambda = \lambda_k^* \) and an optimal cuboid \( R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*} \), so

\[ k \leq N(\lambda_k^*) \leq \left( \frac{\lambda_k^*}{6\pi^2} \right)^{3/2} - \frac{\lambda_k^*}{8\pi a_{1,k}} + \frac{(\lambda_k^*)^{1/2}}{16(a_{1,k}^*)^2}, \]

and, by rearranging, we obtain that

\[ \frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2 \lambda_k^*} \geq \frac{1}{8\pi a_{1,k}} - \frac{(\lambda_k^*)^{-1/2}}{16(a_{1,k}^*)^2}. \]  

(4.6)

The left-hand side of (4.6) is an increasing function of \( \lambda_k^* \), so it is bounded from above by \( \frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \), where \( \nu_k \) is the \( k \)th Dirichlet eigenvalue of the Laplacian on the unit cube in \( \mathbb{R}^3 \). We obtain a lower bound for the right-hand side of (4.6) by using the fact that

\[ \lambda_k^* \geq \lambda_1 R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*} \geq \frac{\pi^2}{(a_{1,k}^*)^2}, \]

implies that

\[ \frac{(\lambda_k^*)^{-1/2}}{16(a_{1,k}^*)^2} \geq \frac{1}{16\pi a_{1,k}^*}. \]  

(4.7)

Hence, by (4.7), we have that

\[ \frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \geq \frac{1}{16\pi a_{1,k}^*}, \]

(4.8)
which implies that,
\[ a_{1,k}^* \geq \frac{1}{16\pi} \frac{6\pi^2 \nu_k}{\nu_k^{3/2} - 6\pi^2 k}. \]  

(4.8)

We now obtain a uniform lower bound for \( a_{1,k}^* \). Let \( \omega_3 \) denote the measure of a ball of radius 1 in \( \mathbb{R}^3 \). Then, by an estimate of Gauss, we have that
\[ N(\nu_k) = \# \left\{ (i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 \leq \frac{\nu_k}{\pi^2} \right\} \geq \frac{\omega_3}{8} \left( \frac{\nu_k^{1/2}}{\pi} - 3^{1/2} \right)^3 \geq \frac{\nu_k^{3/2}}{6\pi^2} - \frac{3^{1/2} \nu_k}{2\pi}. \]

Let \( \Theta_k \) denote the multiplicity of \( \nu_k \). Then \( N(\nu_k) \leq k + \Theta_k - 1 \). In addition, \( \Theta_k = \# \{ (i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 = \frac{\nu_k}{\pi^2} \} \) is the number of integer lattice points in the first quadrant that lie on the sphere in \( \mathbb{R}^3 \) which is centred at \( (0, 0, 0) \) and has radius \( \frac{\nu_k^{1/2}}{\pi} \). By projection onto the plane \( i_3 = 0 \), each of these lattice points corresponds to an integer lattice point which lies inside or on the circle \( \{(i_1, i_2) \in \mathbb{Z}^2 : i_1^2 + i_2^2 = \frac{\nu_k}{\pi^2} \} \) in the first quadrant. The number of integer lattice points which lie inside or on this circle is bounded from above by \( \frac{\nu_k}{4\pi} \), i.e., the area inscribed by the circle in the first quadrant. Thus we obtain that
\[ \nu_k^{3/2} \leq 6\pi^2 k + 3\pi \nu_k \left( \frac{1}{2} + 3^{1/2} \right). \]

Hence by (4.8) and (4.9), we have that
\[ a_{1,k}^* \geq \left( 8 \left( \frac{1}{2} + 3^{1/2} \right) \right)^{-1}. \]

(4.10)

Using that \( a_{1,k}^* \leq a_{2,k}^* \leq a_{3,k}^* \), \( a_{1,k}^* a_{2,k}^* a_{3,k}^* = 1 \) and (4.10), we deduce that
\[ a_{3,k}^* \leq \frac{1}{(a_{1,k}^*)^2} \leq 64 \left( \frac{1}{2} + 3^{1/2} \right)^2 \leq 319. \]

The main obstructions to proving a corresponding result to Theorem 1.1(ii) in higher dimensions \( m \geq 4 \) are the following. Firstly, for \( m \geq 4 \) the corresponding upper bound for \( N(\lambda) \) to (4.2) involves lattice point sums \( \sum_{i=1}^{[R]} g(i) \) with \( g(i) \), \( R \) as in (3.3) and \( n \geq 3 \). For \( n \geq 3 \), \( \frac{\nu_k^{1/2}}{\pi} \) is an inflection point of \( g \) in \( (0, \frac{\nu_k^{1/2}}{\pi}) \) and so \( g \) is not concave on \( (0, \frac{\nu_k^{1/2}}{\pi}) \). Thus, the above approach cannot be used to obtain an upper bound for the left-hand side of (3.1) when \( n \geq 3 \). Secondly, the higher-dimensional equivalent of (4.1) will contain more terms in the right-hand side. The leading term in that right-hand side is the Weyl term. However, the lower order terms are bounds which are uniform in \( a_1 \), for example. Their usefulness depends on the numerical coefficients which show up. These in turn depend on lower dimensional lattice point sums.

5 Proof of Theorem 1.1(ii).

The minimisers \( R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*} \) of \( \lambda_k \) need not be unique. From this point onwards, we consider an arbitrary subsequence of minimisers denoted by \( (R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*})_k \).

For \( E(\lambda) \) as defined in (1.5), we introduce the following notation.
\[
T(\lambda) = \# \{ (x_1, x_2, x_3) \in \mathbb{Z}^3 \cap E(\lambda) \}, \\
T_{x_1}(\lambda) = \# \{ (0, x_2, x_3) \in (\{0\} \times \mathbb{Z}^2) \cap E(\lambda) \}, \\
T_{x_1}^+(\lambda) = \# \{ (0, x_2, x_3) \in (\{0\} \times \mathbb{N}^2) \cap E(\lambda) \}.
\]

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\(T(\lambda)\) is the total number of integer lattice points that are inside or on the ellipsoid \(E(\lambda)\) in \(\mathbb{R}^3\). Similarly \(T_{x_1}(\lambda)\) is the number of integer lattice points that are inside or on the ellipse in \(\mathbb{R}^2\) which is centred at \((0,0)\) and has semi-axes \(\frac{a_1\lambda^{1/2}}{\pi}, \frac{a_2\lambda^{1/2}}{\pi}\). \(T_{x_1}^+(\lambda)\) is the number of these lattice points that lie in the first quadrant (excluding the axes). \(T_{x_2}(\lambda), T_{x_3}(\lambda)\) etc. are defined similarly. Thus, we have that

\[
T(\lambda) = 8N(\lambda) + 4T_{x_1}^+(\lambda) + 4T_{x_2}^+(\lambda) + 4T_{x_3}^+(\lambda)
+ 2 \left[ \frac{a_1\lambda^{1/2}}{\pi} \right] + 2 \left[ \frac{a_2\lambda^{1/2}}{\pi} \right] + 2 \left[ \frac{a_3\lambda^{1/2}}{\pi} \right] + 1,
\]

which implies that

\[
N(\lambda) = \frac{1}{8}T(\lambda) - \frac{1}{2}T_{x_1}^+(\lambda) - \frac{1}{2}T_{x_2}^+(\lambda) - \frac{1}{2}T_{x_3}^+(\lambda)
- \frac{1}{4} \left[ \frac{a_1\lambda^{1/2}}{\pi} \right] - \frac{1}{4} \left[ \frac{a_2\lambda^{1/2}}{\pi} \right] - \frac{1}{4} \left[ \frac{a_3\lambda^{1/2}}{\pi} \right] - \frac{1}{8}.
\]

In addition, we have that

\[
T_{x_1}(\lambda) = 4T_{x_1}^+(\lambda) + 2 \left[ \frac{a_2\lambda^{1/2}}{\pi} \right] + 2 \left[ \frac{a_3\lambda^{1/2}}{\pi} \right] + 1,
\]

which implies that

\[
T_{x_1}^+(\lambda) = \frac{1}{4}T_{x_1}(\lambda) - \frac{1}{2} \left[ \frac{a_2\lambda^{1/2}}{\pi} \right] - \frac{1}{2} \left[ \frac{a_3\lambda^{1/2}}{\pi} \right] - \frac{1}{4},
\]

and similarly for \(T_{x_2}^+(\lambda), T_{x_3}^+(\lambda)\). Thus, we obtain

\[
N(\lambda) = \frac{1}{8}T(\lambda) - \frac{1}{8}T_{x_1}(\lambda) - \frac{1}{8}T_{x_2}(\lambda) - \frac{1}{8}T_{x_3}(\lambda)
+ \frac{1}{4} \left[ \frac{a_1\lambda^{1/2}}{\pi} \right] + \frac{1}{4} \left[ \frac{a_2\lambda^{1/2}}{\pi} \right] + \frac{1}{4} \left[ \frac{a_3\lambda^{1/2}}{\pi} \right] + \frac{1}{4}.
\]

Below we use this expression for \(N(\lambda)\) in order to prove Theorem 1.1(ii).

**Proof of Theorem 1.1(ii).** By setting \(\lambda = \lambda_k^*\) in (5.1) and considering an optimal cuboid \(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}\), we have that

\[
k \leq N(\lambda_k^*) = \frac{1}{8}T(\lambda_k^*) - \frac{1}{8}T_{x_1}(\lambda_k^*) - \frac{1}{8}T_{x_2}(\lambda_k^*) - \frac{1}{8}T_{x_3}(\lambda_k^*)
+ \frac{1}{4} \left[ \frac{a_{1,k}^* (\lambda_k^*)^{1/2}}{\pi} \right] + \frac{1}{4} \left[ \frac{a_{2,k}^* (\lambda_k^*)^{1/2}}{\pi} \right] + \frac{1}{4} \left[ \frac{a_{3,k}^* (\lambda_k^*)^{1/2}}{\pi} \right] + \frac{1}{4}.
\]

By Lemma 4.2, the \(\{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*\}\) are uniformly bounded, so it is possible to make use of known estimates for the number of integer lattice points that are inside or on a 3-dimensional ellipsoid or a 2-dimensional ellipse. In particular there exists \(C < \infty\) such that for all \(\lambda \geq 0\)

\[
\frac{4}{3\pi^2} \lambda^{3/2} - C\lambda^{\beta/2} \leq T(\lambda) \leq \frac{4}{3\pi^2} \lambda^{3/2} + C\lambda^{\beta/2} + 1,
\]

where \(\beta\) is as defined in the Introduction. Similarly there exists \(D < \infty\) such that for all \(\lambda \geq 0\)

\[
\frac{a_2a_3}{\pi} \lambda - D\lambda^{\theta/2} \leq T_{x_1}(\lambda) \leq \frac{a_2a_3}{\pi} \lambda + D\lambda^{\theta/2} + 1,
\]

where \(\theta\) is an exponent of the remainder in Gauss’ circle problem

\[
\# \{(i_1, i_2) \in \mathbb{Z}^2: i_1^2 + i_2^2 \leq R^2\} - \pi R^2 = O(R^\theta), R \to \infty.
\]
The best known estimate to date is $\theta > \frac{131}{208}$, see the Introduction in [16]. Hence the formula above holds for $\theta = \frac{131}{208} + \epsilon$ for any $\epsilon > 0$. The corresponding inequalities to (5.4) also hold for $T_{x_2}(\lambda), T_{x_3}(\lambda)$. Using these inequalities and (5.2), we obtain the following upper bound for $N(\lambda_k^*)$.

\[
k \leq N(\lambda_k^*) \leq \frac{(\lambda_k^*)^{3/2}}{6\pi^2} - \frac{1}{8\pi} \left( \frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*} + \frac{1}{a_{3,k}^*} \right) \lambda_k^* + \frac{C}{8} (\lambda_k^*)^{3/2} + \frac{1}{4\pi} (a_{1,k}^* + a_{2,k}^* + a_{3,k}^*) (\lambda_k^*)^{1/2} + \frac{3D}{8} (\lambda_k^*)^{5/2} + \frac{3}{8}.
\]  

(5.5)

Rearranging (5.5), we obtain that

\[
\frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*} + \frac{1}{a_{3,k}^*} \leq 8\pi \left( \frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \right) + \pi C(\lambda_k^*)^{-2(2-\beta)/2} + 2(\lambda_1^* + \lambda_2^* + \lambda_3^*) (\lambda_k^*)^{-1/2} + 3\pi D(\lambda_k^*)^{(2-\theta)/2} + 3\pi (\lambda_k^*)^{-1}.
\]

Since $\frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k}$ is an increasing function of $\lambda_k^*$, we can replace $\lambda_k^*$ by $\nu_k$, where $\nu_k$ is the $k$th Dirichlet eigenvalue of the Laplacian on the unit cube in $\mathbb{R}^3$. Thus, by Pólya’s Inequality $\lambda_k^* \geq (6\pi^2 k)^{2/3}$, ([19, 20]), we obtain

\[
\frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*} + \frac{1}{a_{3,k}^*} \leq 8\pi \left( \frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \right) + \pi C(\nu_k)^{-2(2-\beta)/2} + 3\pi (\nu_k)^{-1} + 2(\nu_1^* + \nu_2^* + \nu_3^*)(\nu_k)^{-1/2} + 3\pi D(\nu_k)^{(2-\theta)/2} + 3\pi (\nu_k)^{-1}.
\]

(5.6)

To obtain an upper bound for $\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k}$ we proceed as follows. By (5.1) with $\lambda = \nu_k$ we have that

\[
N(\nu_k) = \frac{1}{8} T(\nu_k) - \frac{3}{8} T_{x_1}(\nu_k) + \frac{3}{4} \left[ \frac{\nu_k^{1/2}}{\pi} \right] + \frac{1}{4}.
\]

(5.7)

Since $a_1 = a_2 = a_3 = 1$, by (5.3) and (5.4), we have that

\[
\frac{4}{3\pi^2} \nu_k^{3/2} - C\nu_k^{\beta/2} \leq T(\nu_k),
\]

(5.8)

and

\[
T_{x_1}(\nu_k) \leq \frac{\nu_k}{\pi} + D\nu_k^{\theta/2} + 1,
\]

(5.9)

where $\beta$ and $\theta$ are as in (5.3), (5.4). Again let $\Theta_k$ denote the multiplicity of $\nu_k$. Thus by (5.7), (5.8) and (5.9), we obtain a lower bound for $N(\nu_k)$:

\[
k + \Theta_k - 1 \geq N(\nu_k) \geq \frac{\nu_k^{3/2}}{6\pi^2} - \frac{C}{8} \nu_k^{\beta/2} - \frac{3\nu_k - 3D}{8} \nu_k^{\theta/2} + \frac{1}{4\pi} \nu_k^{1/2} - \frac{7}{8},
\]

which implies that

\[
\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \leq \frac{3}{8\pi} + C \frac{\nu_k^{-(2-\beta)/2}}{8} + \frac{3D}{8} \nu_k^{-(2-\theta)/2} - \frac{3}{4\pi} \nu_k^{-1/2} + \Theta_k \nu_k^{-1} - \frac{1}{8} \nu_k^{-1}
\]

\[
\leq \frac{3}{8\pi} + C \frac{\nu_k^{-(2-\beta)/2}}{8} + \frac{3D}{8} \nu_k^{-(2-\theta)/2} + \Theta_k \nu_k^{-1}
\]

\[
\leq \frac{3}{8\pi} + C \frac{(6\pi^2)^{-(2-\beta)/3} k^{-(2-\beta)/3}}{8} + \frac{3D}{8} (6\pi^2)^{-(2-\theta)/3} k^{-(2-\theta)/3} + \Theta_k \nu_k^{-1},
\]

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by Pólya’s Inequality.

We have that \( \Theta_k = \# \{ (i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 = \frac{4k}{\pi} \} \) is the number of integer lattice points in the first octant that lie on the sphere in \( \mathbb{R}^3 \) which is centred at \((0,0,0)\) and has radius \( \frac{4k}{\pi} \). It is well known that \( \# \{ (x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d \} = O(d^{1/2 + o(1)}) \).

The following routine proof was communicated by T. Wooley. Let \( n = d - x_3^2 \). Now \( |x_3| \leq d^{1/2} \), so for \( x_3 \in [-d^{1/2}, d^{1/2}] \cap \mathbb{Z} \), there are at most \( 2d^{1/2} + 1 \) possible values of \( n \). If \( n = 0 \), then \( x_1^2 + x_2^2 = 0 \) has one solution \((0,0) \in \mathbb{Z}^2 \). Suppose that \( n \neq 0 \). Let \( R(n) \) denote the number of pairs \((x_1, x_2) \in \mathbb{Z}^2 \) such that \( x_1^2 + x_2^2 = n \). Then

\[
\# \{ (x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d \} \leq 1 + \sum_{|z| \leq d^{1/2}} R(d - z^2).
\]

By Corollary 3.23 of [18], we have that

\[
R(n) = 4 \sum_{d|n, d > 0, d \text{ odd}} \left( \frac{-1}{d} \right),
\]

where the sum is taken over all positive, odd divisors of \( n \) and \( \left( \frac{-1}{d} \right) \) is the quadratic residue symbol. Thus \( R(n) \leq 4D(n) \), where \( D(n) \) denotes the number of positive divisors of \( n \). By Theorem 8.31 of [18], for every \( \epsilon > 0 \), there exists \( n_\epsilon \) such that for \( n > n_\epsilon \),

\[
D(n) < n^{(1+\epsilon)\log 2/\log \log n},
\]

which implies that \( D(n) = O(n^\epsilon) \). Therefore we obtain that

\[
\# \{ (x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d \} \leq 1 + O\left( \sum_{|z| \leq d^{1/2}} (d - z^2)^\epsilon \right) \leq 1 + O(d^{1/2+\epsilon}).
\]

So \( \Theta_k = O(\nu_k^{3/2+o(1)}) \) and \( \Theta_k \nu_k^{-1} = O(\nu_k^{-1/2+o(1)}) = O(k^{-1/2+o(1)}) \). Thus we obtain

\[
\frac{\nu_k^{3/2} - 6\pi^2k}{6\pi^2\nu_k} \leq \frac{3}{8\pi} + O(k^{-(2-\beta)/3}). \tag{5.10}
\]

So by (5.6) and (5.10), we deduce that

\[
\frac{1}{a_{1,k}} + \frac{1}{a_{2,k}} + \frac{1}{a_{3,k}} \leq 3 + O(k^{-(2-\beta)/3}), \quad k \to \infty. \tag{5.11}
\]

Furthermore, by the Arithmetic Mean – Geometric Mean Inequality applied to \( \frac{1}{a_{1,k}} + \frac{1}{a_{2,k}} \), we have by (5.11) that

\[
2(a_{3,k}^*)^{1/2} + \frac{1}{a_{3,k}} \leq 3 + O(k^{-(2-\beta)/3}), \quad k \to \infty.
\]

Let \( a_{3,k}^* = 1 + \delta_k \) where \( \delta_k > 0 \). Then

\[
2(1 + \delta_k)^{3/2} + 1 \leq 3 + 3\delta_k + O(k^{-(2-\beta)/3}), \quad k \to \infty.
\]

Since \( a_{3,k}^* \leq 319, \delta_k \leq 399 \). Hence \((1 + \delta_k)^{3/2} \geq 1 + \frac{3}{2}\delta_k + \frac{3}{1600}\delta_k^2 \) for \( 0 < \delta_k \leq 399 \), we deduce that \( \delta_k \leq O(k^{-(2-\beta)/6}), k \to \infty \). As this estimate is independent of the subsequence \( (R_{a_1,k,a_2,k,a_3,k})_k \), we arrive at the conclusion of Theorem 1.1(ii).\]
References


